For Online Publication: Appendix for Government Policy with Time Inconsistent Voters

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Abstract

In this supplementary appendix, we provide proofs of claims and propositions pertaining to the setting with arbitrary number of periods and convex distortions discussed in the body of the paper. We also analyze an analogous model to that presented in the paper in which agents are characterized by Gul and Pesendorfer (2004) preferences, as well as a model in which consumption is possible in period 1, in addition to periods 2 and 3.

1 Arbitrary Number of Periods – Proofs

Proof of Proposition 8

The proof proceeds as follows. First of all we derive first order conditions of the two maximization problems discussed in the text:

- The investment problem of any agent at time $t = 1$ choosing the sequence of period 1 savings in illiquid assets $\{s_{1t} \geq 0\}_{t=1}^{T}$, for any given deficit and repayment sequence $\{d_t, q_t\}_{t=2}^{T}$:

$$\begin{array}{l} \max\ u(s_{11}) + \beta \sum_{t=2}^{T}\ u(s_{1t} + d_t - A\left(q_t\right)) \\ s.t.\quad \sum_{t=1}^{T}\ s_{1t} = k; \end{array} \tag{I}$$

- The political economy problem at any election at time $t \geq 2$, reduced to the choice of maps $d_t(D_{t-1}),\ q_t(D_{t-1}) \geq 0$, for given time 1 transfers $\{s_{1\tau}\}_{\tau=t}^{T}$ and given expected future maps $d_\tau(D_{\tau-1}), q_\tau(D_{\tau-1})$ for all $t+1 \leq \tau \leq T$:

$$\begin{array}{l} \max\ u\left(s_{1t} + d_t(D_{t-1}) - A\left(q_\tau(D_{\tau-1})\right)\right) + \beta \sum_{\tau=t+1}^{T}\ u\left(s_{1\tau} + d_\tau(D_{\tau-1}) - A\left(q_\tau(D_{\tau-1})\right)\right) \\ s.t.\quad \sum_{t=2}^{T}\ d_t - q_t = 0. \end{array} \tag{PE}$$

We then derive several implications, notably regarding the structure of the debt accumulation and repayment phases at equilibrium. To this end we exploit the condition that k is large enough, but we obtain as a by-product a characterization of the structure of equilibria when the condition is not imposed. Finally, we derive properties of the consumption sequence at equilibrium.

Recall the government's budget balance, the constraint in Problem PE is:

$$\sum_{t=2}^{T} d_t - q_t = 0. \tag{1}$$

By definition, $D_t = \sum_{\tau=2}^{t} d_\tau - q_\tau$. It then follows that (1) can also be written as $D_t + \sum_{\tau=t+1}^{T} d_\tau - q_\tau = 0$, for any $t \geq 2$; which in turn implies, for $t = T$, $D_T = 0$. Furthermore, using again the definition of D_t and taking derivatives, $dD_t = dd_t$ and $dD_t = -dq_t$. Let $J_q(\tau)$ (respectively $J_d(\tau)$) denote the subset of periods $j > \tau$ such that $q_j > 0$ (respectively. $d_j \geq 0$ with $q_t = 0$). Therefore, government's budget balance, (1), implies

$$\sum_{j \in J_q(\tau)} \frac{\partial q_j}{\partial D_\tau} - \sum_{j \in J_d(\tau)} \frac{\partial d_j}{\partial D_\tau} = 1. \tag{2}$$

Notice that, at equilibrium, $d_t > 0$ for some $2 \leq t \leq T$. This can be shown by contradiction and along the lines of Proposition 2. Government budget balance, equation (1), implies that $q_\tau > 0$ for some $2 \leq \tau \leq T$. Consider a period $2 \leq \tau < T$ such that $q_\tau > 0$. The first order condition of Problem PE at τ is:

$$0 = A'(q_\tau)u'(s_{1\tau} - q_\tau) - \beta \left[\sum_{j \in J_q(\tau)} A'(q_j)u'(c_j)\frac{\partial q_j}{\partial D_\tau} - \sum_{j \in J_d(\tau)} u'(c_j)\frac{\partial d_j}{\partial D_\tau} \right] \tag{3}$$

Consider instead a period $2 \leq t < T$ such that $d_t \geq 0$, with $q_t = 0$. The first order condition of Problem PE at t is:

$$0 = u'(d_t + s_{1t}) - \beta \left[\sum_{j \in J_q(t)} A'(q_j)u'(c_j)\frac{\partial q_j}{\partial D_\tau} - \sum_{j \in J_d(t)} u'(c_j)\frac{\partial d_j}{\partial D_\tau} \right] \tag{4}$$

Recall that, by the implications of (1) derived above, $\sum_{j \in J_q(t)} \frac{\partial q_j}{\partial D_\tau} - \sum_{j \in J_d(t)} \frac{\partial d_j}{\partial D_\tau} = 1$. Furthermore, it can be shown that the first order conditions of problem PE imply that

$$\frac{\partial q_j}{\partial D_\tau} > 0 \text{ and } \frac{\partial d_j}{\partial D_\tau} < 0, \quad \text{for all } j > \tau. \tag{5}$$

This is a consequence of consumption smoothing and can be formally shown by deriving envelope conditions from (4).

Notice that the solution of Problem I requires

$$u'(d_j + s_{1j}) \leq u'(s_{1j'} - q_{j'}) \text{ for all } j \in J_d(1),\ j' \in J_q(1), \tag{6}$$

with equality for all j, j' such that $s_{1j}, s_{1,j'} > 0$ (that is, when the solution of Problem I is interior). As a consequence, in particular, the solution of Problem I requires that $u'(c_j)$ be constant for all j such that $q_j > 0$, that is for $j \in J_q(1)$. This is so because, by Inada conditions, $q_j > 0$ implies $s_{1j} > 0$.

Conditions (3) and (4) allow us to characterize the structure of the debt accumulation and repayment phases at equilibrium. We show that i) $q_T > 0$ and that ii) $q_\tau > 0$ implies that $q_j > 0$ for all $j > \tau$. To prove *i)* we proceed by contradiction, postulating that $d_T \geq 0$ with $q_T = 0$. Consider first the case in which $q_{T-1} > 0$. Then 3) implies

$$A'(q_{T-1})u'(c_{T-1}) = \beta u'(d_T + s_{1T})$$

But $q_{T-1} > 0$ implies that $A'(q_{T-1}) > 1$, while $\beta < 1$. As a consequence, the condition cannot be satisfied as it requires $u'(c_{T-1} < u'(d_T + s_{1T})$, which is in contradiction with (6) and hence with the solution of Problem I. The same logic applies to any candidate equilibrium characterized by an uninterrupted sequence of $d_j \geq 0$, from some t up to T and $q_{t-1} > 0$. We conclude $q_T > 0$. The proof of ii) also runs by contradiction, postulating that $q_\tau > 0$ and $d_j \geq 0$ with $q_j = 0$, for some $j > \tau$ (recall that $d_t q_t = 0$ and hence $d_t > 0$ implies $q_t = 0$). Consider first the case that $q_{T-2} > 0$, and $d_{T-1} \geq 0$ with $q_{T-1} = 0$. Recall we have just shown that $q_T > 0$. Then (3) implies

$$\begin{aligned}
A'(q_{T-2})u'(c_{T-2}) &= \beta\left[A'(q_T)u'(c_T)\frac{\partial q_T}{\partial D_{T-2}} - u'(d_{T-1} + s_{1T-1})\frac{\partial d_{T-1}}{\partial D_{T-2}}\right] \\
u'(d_{T-1} + s_{1T-1}) &= \beta A'(q_T)u'(c_T)
\end{aligned}$$

But $q_{T-2} > 0$, $q_T > 0$ imply $A'(q_{T-2})$, $A'(q_T) > 0$ and $u'(c_{T-2}) = u'(c_T)$ as an implication of (6). Furthermore, Using (5), the first equation can then be written as $A'(q_{T-2}) = \beta\left[A'(q_T)\frac{\partial q_T}{\partial D_{T-2}} + \frac{u'(c_{T-1})}{u'(c_T)} \mid \frac{\partial d_{T-1}}{\partial D_{T-2}} \mid\right]$ and by government's budget balance, equation (1), $\frac{\partial q_T}{\partial D_{T-2}} + \mid \frac{\partial d_{T-1}}{\partial D_{T-2}} \mid = 1$. Also, $d_{T-1} \geq 0$ with $q_{T-1}) = 0$ implies $\frac{u'(c_{T-1})}{u'(c_T)} \leq 1$ by (6). As a consequence, the first equation implies $\beta A'(q_T) > 1$, which when substituted into the second requires $\frac{u'(c_{T-1})}{u'(c_T)} > 1$, a contradiction with (6). The same logic applies to any candidate equilibrium such that $q_\tau > 0$ is followed at some $t > \tau$ by $q_t = 0$ with $d_t \geq 0$. We conclude that $q_\tau > 0$ implies that $q_j > 0$ for all $j > \tau$.

We now show that, for k large enough, $\beta A'(q_T) > 1$ (recall that $q_T > 0$). Suppose on the contrary that $\beta A'(q_T) \leq 1$. Conditions (3) and (4) then imply that $d_{T-1} > 0$ and:

$$u'(d_{T-1}) = \beta A'(q_T) u'(c_T).^1 \tag{7}$$

It is straightforward to show that Problem I implies that, at equilibrium, c_T must increase without bound with the total size of the economy, k. Then, keeping $\beta A'(q_T)$ bounded above by 1, (7) implies that d_{T-1} also increases unboundedly with k. But $q_T \geq d_{T-1}$ by (1) and hence, for k large enough, it must be that $\beta A'(q_T) > 1$, the desired contradiction.

Thus, we only need to consider the case in which $\beta A'(q_T) > 1$. In this case, conditions (3) and (4) imply that $q_{T-1} > 0$. Indeed the solution of Problem PE involves then repayments $q_\tau > 0$ for any $\tau \leq T$ greater than some $\tilde{t} \geq 2$. In this case, condition (3), the first order condition of Problem PE, takes the form:

$$0 = A'(q_\tau) u'(s_{1\tau} - q_\tau) - \beta \left[\sum_{j=\tau+1}^{T} A'(q_j) u'(c_j) \frac{\partial q_j}{\partial D_\tau} \right]$$

Furthermore, (2) reduces to $\sum_{j=\tau+1}^{T} \frac{\partial q_j}{\partial D_\tau} = 1$. But, if $q_\tau > 0$ for any τ greater than some $\tilde{t} \geq 2$, the first order conditions corresponding to the agent's optimization at time $t = 1$ are interior and $s_{1j} > 0$, for any $\tau \leq j \leq T$. The implication of (6) that we derived above then implies that $u'(c_j)$ is constant for any $j \geq \tau$. As a consequence c_j as well as $c_j + \frac{\partial q_j}{\partial D_\tau}$ are constant in j and so is $\frac{\partial q_j}{\partial D_\tau}$. In particular, then, $\frac{\partial q_\tau}{\partial D_\tau} = \frac{1}{T-\tau}$. Summing up, the first order conditions of Problem PE are reduced to

$$A'(q_\tau) = \frac{\beta}{T-\tau} \left[\sum_{j=\tau+1}^{T} A'(q_j) \right]. \tag{8}$$

This ends our proof of part 1 in the statement of Proposition 8.

Having shown that, for k large enough, the dynamics of debt has an accumulation phase followed by a re-payment phase, and having characterized the equilibrium conditions of the repayment phase, we now study the debt accumulation phase. Let $\tilde{t}$ denote the last time τ such that deficit is strictly positive. We now show that the equilibrium condition at $\tilde{t}$ is interior and hence $u'(c_{\tilde{t}}) = u'(c_\tau)$, for any $\tilde{t} < \tau \leq T$. We proceed by contradiction. We have shown that a repayment $q_\tau > 0$ occurs when expected future marginal distortions $\frac{\beta}{T-\tau} \left[\sum_{j=\tau+1}^{T} A'(q_j) \right] > 1$. At $\tilde{t}$ then:

[1] Indeed, this argument implies that, at equilibrium, $q_t = 0$ and $d_t > 0$ for any $2 \leq t \leq T-1$: debt is accumulated until period $T-1$ and repayed at time T.

$$\frac{\beta}{T-\tilde{t}}\left[\sum_{j=\tilde{t}+1}^{T} A'(q_j)\right] \leq 1.$$

Assume by way of contradiction that $\frac{\beta}{T-\tilde{t}}\left[\sum_{j=\tilde{t}+1}^{T} A'(q_j)\right] < 1$. This implies $\frac{\beta}{T-\tilde{t}}A'(q_{\tilde{t}+1}) < 1 - \frac{\beta}{T-\tilde{t}}\left[\sum_{j=\tilde{t}+2}^{T} A'(q_j)\right]$. But since $q_{\tilde{t}+1} > 0$ by assumption, the first order conditions at $\tilde{t}+1$ imply $\frac{\beta}{T-\tilde{t}}\left[\sum_{j=\tilde{t}+2}^{T} A'(q_j)\right] > 1$, and hence $\frac{\beta}{T-\tilde{t}}A'(q_{\tilde{t}+1}) < 0$ which is impossible. We can conclude then that

$$\frac{\beta}{T-\tilde{t}}\left[\sum_{j=\tilde{t}+1}^{T} A'(q_j)\right] = 1 \tag{9}$$

which implies $u'(d_{\tilde{t}}) = u'(c_{\tilde{t}+1})$.[2]

We now study the debt accumulation phase up to period $\tilde{t}$. Consider the first order conditions for Problem PE at $\tilde{t}-1$, equation (4). In the accumulation phase these are reduced to:

$$0 = u'(s_{1\tilde{t}-1} + d_{\tilde{t}-1}) + \beta u'(s_{1\tilde{t}} + d_{\tilde{t}})\left[\frac{\partial d_{\tilde{t}}}{\partial D_{\tilde{t}-1}} + \frac{\beta}{T-t}\left[\sum_{\tau=t}^{T} A'(q_\tau)\right]\frac{\partial q_\tau}{\partial D_{\tilde{t}-1}}\right].$$

But the (interior) first order conditions of the agent optimization choice at time $t=1$ implies that $c_{\tilde{t}}$ is constant and hence $D_{\tilde{t}}$ is also constant:

$$\frac{\partial d_{\tilde{t}}}{\partial D_{\tilde{t}-1}} = -1 \text{ and } \frac{\partial q_\tau}{\partial D_{\tilde{t}-1}} = 0, \text{ for any } \tau > \tilde{t}$$

It follows then that the first order conditions in Problem I must hold at the corner $s_{\tilde{t}-1} = 0$:

$$u'(d_{\tilde{t}-1}) = \beta u'(s_{1\tilde{t}} + d_{\tilde{t}})$$

The argument can be extended backwards to imply that $c_\tau = d_\tau$, for any $2 \leq \tau \leq \tilde{t}-1$. Consider period $\tilde{t}-2$. Using again the fact that the (interior) first order conditions of the agent optimization choice at time $t=1$ imply that $c_{\tilde{t}}$ is constant and hence $D_{\tilde{t}}$ is also constant, the first order condition of Problem PE are reduced to:

$$0 = u'(s_{1\tilde{t}-2} + d_{\tilde{t}-2}) + \beta\left[u'(d_{\tilde{t}-1})\frac{\partial d_{\tilde{t}-1}}{\partial D_{\tilde{t}-2}} + u'(s_{1\tilde{t}} + d_{\tilde{t}})\frac{\partial d_{\tilde{t}}}{\partial D_{\tilde{t}-2}}\right]$$

[2] In other words, if $u'(d_{\tilde{t}}) < u'(c_{\tilde{t}+1})$ then in fact the equilibrium will have an extra period of debt accumulation; that is, the last period of debt accumulation will in fact be $\tilde{t}+1$. As a consequence, note that a corner solution with $s_{1\tilde{t}} = 0$ can in fact occur in the $T=3$ economy, in which debt is necessarily accumulated at $t=2$ and there cannot be an extra period of accumulation as repayment must occur at $t=3$; see the analysis of this case in the text.

Substituting $u'(d_{\tilde{t}-1}) = \beta u'(s_{1\tilde{t}} + d_{\tilde{t}})$ we have

$$0 = u'(s_{1\tilde{t}-2} + d_{\tilde{t}-2}) + \beta u'(s_{1\tilde{t}} + d_{\tilde{t}}) \left[\beta \frac{\partial d_{\tilde{t}-1}}{\partial D_{\tilde{t}-2}} + \frac{\partial d_{\tilde{t}}}{\partial D_{\tilde{t}-2}}\right]$$

But $\left[\frac{\partial d_{\tilde{t}-1}}{\partial D_{\tilde{t}-2}} + \frac{\partial d_{\tilde{t}}}{\partial D_{\tilde{t}-2}}\right] = -1$; and hence (5) implies that $\mid \left[\beta \frac{\partial d_{\tilde{t}-1}}{\partial D_{\tilde{t}-2}} + \frac{\partial d_{\tilde{t}}}{\partial D_{\tilde{t}-2}}\right] \mid < 1$. Then again the first order conditions of Problem I must hold at the corner $s_{\tilde{t}-2} = 0$. Furthermore, using the first order conditions we obtained at $\tilde{t} - 2$ and $\tilde{t} - 1$ it follows directly by concavity, using $\mid \left[\beta \frac{\partial d_{\tilde{t}-1}}{\partial D_{\tilde{t}-2}} + \frac{\partial d_{\tilde{t}}}{\partial D_{\tilde{t}-2}}\right] \mid < 1$, that $d_{\tilde{t}-2} > d_{\tilde{t}-1}$. Proceeding recursively back in time, more generally for any $2 \leq \tilde{t} - 1$, the first order conditions will have a similar structure:

$$0 = u'(s_{1t} + d_t) + \beta u'(s_{1\tilde{t}} + d_{\tilde{t}}) \left[\sum_{j=t+1}^{\tilde{t}} \epsilon_j \frac{\partial d_j}{\partial D_t}\right] \tag{10}$$

for some $0 \leq \epsilon_j \leq 1$ such that $|: \sum_{j=t+1}^{\tilde{t}} \epsilon_j \frac{\partial d_j}{\partial D_t} :| :< 1$. As a consequence, $c_\tau = d_\tau$, for any $2 \leq \tau \leq \tilde{t} - 1$. This ends our proof of part 2 in the statement of the proposition. ■

Proof of Corollary 9

In the proof of Proposition 8 we have shown that the first order condition of Problem PE, in the repayment phase, are reduced to (8). We now show that $A'(q_t)$ is an increasing sequence in t. To this end it is sufficient to write (8) recursively as follows:

$$\begin{aligned} A'(q_{T-1}) &= \beta A'(q_T), \\ A'(q_{T-2}) &= \frac{\beta}{2}(1+\beta) A'(q_T), \\ A'(q_{T-3}) &= \frac{\beta}{3}(1+\beta)\left(1+\frac{\beta}{2}\right) A'(q_T), \\ &\dots \\ A'(q_t) &= \frac{\beta}{T-t} \prod_{j=1}^{T-t-1} \left(1+\frac{\beta}{j}\right) A'(q_T). \end{aligned}$$

It can now be directly checked that the sequence $\frac{\beta}{T-t} \prod_{j=1}^{T-t-1} \left(1 + \frac{\beta}{j}\right)$ is increasing in t for given T.

The sequence $A'(q_t)$ is then increasing in t and so is the sequence q_t, as $A(q)$ is strictly convex. Furthermore, as $q_t > 0$ in the repayment phase, s_{1t} must also be, by Inada conditions. The solution of Problem I is then interior, which implies that consumption is equalized across time. Finally, in the proof of Proposition 8 we have shown that at $\tilde{t}$ distortions must satisfy

(9) and that the solution of Problem I is interior at $\tilde{t}$. This implies that $c_{\tilde{t}} = c_{\tilde{t}+1}$. But, by our previous characterization of the repayment phase in this corollary, $c_{\tilde{t}+1} = c_{\tau}$, for any $\tau > \tilde{t} + 1$.

We now show that d_t is decreasing over time in the debt accumulation phase. To this end we need to show that the absolute value of the expression $\left[\sum_{j=t+1}^{\tilde{t}} \alpha_j \frac{\partial d_j}{\partial D_t}\right]$ in equation (10) is increasing in t.

We first establish that $\frac{\partial d_j}{\partial D_t}$ change by the same factor for any j when t changes:

$$\frac{\frac{\partial d_j}{\partial D_t}}{\frac{\partial d_{j'}}{\partial D_t}} = \frac{\frac{\partial d_j}{\partial D_{t'}}}{\frac{\partial d_{j'}}{\partial D_{t'}}},\ 2 < j, j' \leq \tilde{t},\ 2 \leq k < \min\{j, j'\} \tag{11}$$

Indeed, consider the first order condition at time $\tilde{t} - 1$: $u'(c_{\tilde{t}-1}) = \beta u'(c_{\tilde{t}})$. Differentiating, the Envelope Theorem implies,

$$u''(c_{\tilde{t}-1}) \frac{\partial d_{\tilde{t}-1}}{\partial D_{\tilde{t}-2}} = \beta u''(c_{\tilde{t}}) \frac{\partial d_{\tilde{t}}}{\partial D_{\tilde{t}-2}};$$

but also that

$$u''(c_{\tilde{t}-1}) \frac{\partial d_{\tilde{t}-1}}{\partial D_{\tilde{t}-3}} = \beta u''(c_{\tilde{t}}) \frac{\partial d_{\tilde{t}}}{\partial D_{\tilde{t}-3}}.$$

It follows that $\frac{\frac{\partial d_{\tilde{t}}}{\partial D_{\tilde{t}-2}}}{\frac{\partial d_{\tilde{t}-1}}{\partial D_{\tilde{t}-2}}} = \frac{\frac{\partial d_{\tilde{t}}}{\partial D_{\tilde{t}-3}}}{\frac{\partial d_{\tilde{t}-1}}{\partial D_{\tilde{t}-3}}}$. It is straightforward to see that in fact the argument holds for any $2 \leq k < \tilde{t} - 1$. Furthermore, the same logic can be repeated on the first order condition at time $\tilde{t} - 2$. In fact, after differentiating and recalling that, for any $\tau > t$, $\frac{\partial d_{tau}}{\partial D_t} < 0$ by (5), we obtain:

$$u''(c_{\tilde{t}-2}) \frac{\partial d_{\tilde{t}-2}}{\partial D_{\tilde{t}-k}} = \beta u''(c_{\tilde{t}}) \mid \beta \frac{\partial d_{\tilde{t}}}{\partial D_{\tilde{t}-2}} + \frac{\partial d_{t\tilde{-}1}}{\partial D_{\tilde{t}-2}} \mid \frac{\partial d_{\tilde{t}}}{\partial D_{\tilde{t}-k}};$$

and hence $\frac{\frac{\partial d_{\tilde{t}}}{\partial D_{\tilde{t}-k}}}{\frac{\partial d_{\tilde{t}-2}}{\partial D_{\tilde{t}-k}}}$ is constant in k. Once again, the same argument holds for $\tilde{t} - 3$, $\tilde{t} - 4$ and so on backwards until period 2.

Simplify notation by letting $\mid \sum_{j=t+1}^{\tilde{t}} \epsilon_j \frac{\partial d_j}{\partial D_t} \mid$ be denoted Γ_t. Then, developing first order conditions backwards from $\tilde{t} - 1$ we have:

$$\begin{aligned}
\Gamma_{\tilde{t}-1} &= \beta \\
\Gamma_{\tilde{t}-2} &= \mid \beta \frac{\partial d_{\tilde{t}-1}}{\partial D_{\tilde{t}-2}} + \frac{\partial d_{\tilde{t}}}{\partial D_{\tilde{t}-2}} \mid \\
\Gamma_{\tilde{t}-3} &= \mid \beta \Gamma_{\tilde{t}-2} \frac{\partial d_{\tilde{t}-2}}{\partial D_{\tilde{t}-3}} + \beta \frac{\partial d_{\tilde{t}-1}}{\partial D_{\tilde{t}-3}} + \frac{\partial d_{\tilde{t}}}{\partial D_{\tilde{t}-3}} \mid \\
\Gamma_{\tilde{t}-4} &= \mid \beta \Gamma_{\tilde{t}-3} \frac{\partial d_{\tilde{t}-3}}{\partial D_{\tilde{t}-4}} + \Gamma_{\tilde{t}-2} \frac{\partial d_{\tilde{t}-2}}{\partial D_{\tilde{t}-3}} + \Gamma_{\tilde{t}-2} \frac{\partial d_{\tilde{t}-2}}{\partial D_{\tilde{t}-4}} + \beta \frac{\partial d_{\tilde{t}-1}}{\partial D_{\tilde{t}-4}} + \frac{\partial d_{\tilde{t}}}{\partial D_{\tilde{t}-4}} \mid \\
&\quad .....
\end{aligned}$$

Using (11), however the sequence of first order conditions can be written as follows:

$$
\begin{aligned}
\Gamma_{\tilde{t}-1} &= \beta \\
\Gamma_{\tilde{t}-2} &= |: \beta \frac{\partial d_{\tilde{t}-1}}{\partial D_{\tilde{t}-2}} + \frac{\partial d_{\tilde{t}}}{\partial D_{\tilde{t}-2}} \quad | \\
\Gamma_{\tilde{t}-3} &= \beta \Gamma_{\tilde{t}-2} \mid \frac{\partial d_{\tilde{t}-2}}{\partial D_{\tilde{t}-3}} + \Gamma_{\tilde{t}-2} \left(1 - \mid \frac{\partial d_{\tilde{t}-2}}{\partial D_{\tilde{t}-3}} \mid \right) \\
\Gamma_{\tilde{t}-4} &= \beta \Gamma_{\tilde{t}-3} \mid \frac{\partial d_{\tilde{t}-3}}{\partial D_{\tilde{t}-4}} \mid + \Gamma_{\tilde{t}-3} \left(1 - \mid \frac{\partial d_{\tilde{t}-3}}{\partial D_{\tilde{t}-4}} \mid \right) \\
&\quad .....
\end{aligned}
$$

and hence Γ_t is increasing in t. ■

Proof of Corollary 10

Consider an economy with aggregate endowment k and T periods such that the characterization in Proposition 8 holds. Now consider replicas of this economy characterized by aggregate endowment ρk and ρT periods, for some $\rho > 1$. Let $\tilde{t}(\rho)$ denote the last accumulation period at the equilibrium of the replica economy; $\tilde{t}(1)$ is then the last accumulation period in the original economy with endowment k and T periods. Let $c_t(\rho)$ (respectively. $c_t(1)$) denote the consumption at period t in the replica economy (respectively. in the original economy). We show that the maximal debt of any replica ρ increases with respect to the original economy, $D_{\tilde{t}(\rho)} > D_{\tilde{t}(1)}$. As a consequence, the sequence $D_{\tilde{t}(\rho)}$ increases in ρ. The proof proceeds by contradiction. Assume by way of contradiction that $D_{\tilde{t}(\rho)} \leq D_{\tilde{t}(1)}$. Consider first the case in which $D_{\tilde{t}(\rho)} = D_{\tilde{t}(1)}$ and the sequence of repayments is unchanged, satisfying (8). Note that in this case, as the characterization in Proposition 8 holds, consumption is constant at equilibrium from the last accumulation period up to the last period (hence over the repayment period). Therefore, we must have

$$
\begin{aligned}
\tilde{t}(\rho) &= T + \tilde{t}(1) \\
c_t(1) &= \frac{k - c_1(1)}{\tilde{t}(1) + 1}, \quad \text{for any } \tilde{t}(1) \leq t \leq T \\
c_{T+t}(\rho) &= \frac{2k - c_1(\rho)}{\tilde{t}(1) + 1}, \quad \text{for any } \tilde{t}(1) \leq t \leq T
\end{aligned}
$$

From the first order condition of Problem I it can be shown that, while $c_1(\rho) > c_1(1)$, $c_{T+t}(\rho) > c_t(1)$, for any $t > \tilde{t}(1)$. As a consequence, comparing the first order condition of

Problem PE in the replica economy at at $T+\tilde{t}(1)-1$ with that of of the original economy at $\tilde{t}(1)-1$ implies that the deficit in the replica economy is higher than in the original economy. Solving backwards the first order condition of Problem PE we have that the maximal debt accumulated in the replica economy must be higher than in the original economy, $D_{\tilde{t}(\rho)} > D_{\tilde{t}(1)}$ yielding the desired contradiction. Note that, as a consequence of equations (8) and (9), if $D_{\tilde{t}(\rho)} = D_{\tilde{t}(1)}$ the sequence of repayments must indeed be unchanged. A similar argument can be applied to the case in which $D_{\tilde{t}(\rho)} < D_{\tilde{t}(1)}$. In this case in fact the repayment phase still needs to satisfy equations (8) and (9). As a consequence, if $D_{\tilde{t}(\rho)} < D_{\tilde{t}(1)}$ the repayment phase is possibly shorter. A fortiori then $c_{T+t}(\rho) > c_t(1)$ for any t such that $T+t$ is in the repayment phase of the replica economy. Comparing the first order condition of Problem PE in the replica economy at at $\tilde{t}(\rho)-1$ with that of of the original economy at $\tilde{t}(1)-1$ and solving backwards the first order condition of Problem PE produces a contradiction, as in the previous case.

We conclude that along the sequence of replica economies the maximal debt accumulated must be increasing, $D_{\tilde{t}(\rho)}$ increases with ρ. In fact, equations (8) and (9) imply that the repayment phase $\rho T-\tilde{t}(\rho)$ must also be increasing. But the sequence of maximal debt cannot have an upper bound. If it did, the sequence $\tilde{t}(\rho)$ would be bounded and the length of the repayment phase would instead grow to infinity, $\rho T-\tilde{t}(\rho)\to\infty$. This is not possible. In fact, in the proof of Corollary 9 we have shown that the repayment phase can be alternatively characterized solving (8) recursively. Proceeding along these lines we obtain

$$\lim_{\rho\to\infty}\frac{\beta}{\rho T-\tilde{t}(\rho)}\prod_{j=1}^{\rho T-\tilde{t}(\rho)-1}\left(1+\frac{\beta}{j}\right)=0 \text{ if } \lim_{\rho\to\infty}\left[\rho T-\tilde{t}(\rho)\right]\to\infty$$

This can be shown by applying the ratio convergence test (after a log transformation). As a consequence, $A'(q_{\tilde{t}(\rho)})\to 0$ as $\rho\to\infty$. In other words, the right-hand-side of equation (9) converges to 0 as $\rho\to\infty$, violating equation (9) itself. ■

2 Gul-Pesendorfer Preferences

The paper provides an analysis of voters who are characterized by quasi-hyperbolic preferences. One could also contemplate a setting in which agents experience temptation costs in each period a-la Gul and Pesendorfer (2001, 2004). In this Section, we show that the underlying forces driving our results do not change in such an alternative modeling setup. Indeed, suppose that, as in Gul and Pesendorfer two functions u and v govern an individual's valuations of choices from a set X. We adopt the assumption on temptation in Gul and Pesendorfer (2004), i.e, temptation in period t is given by the option of consuming the maximal feasible amount in period t. To their model we first introduce the possibility of illiquid assets and then add government debt.

As in the baseline model used in the paper, there is a wealth k and three periods. In period 1 the agent does not consume but just saves for subsequent periods. If there is no access to illiquid assets, and therefore, no possibility of commitment, in period 1 the agent can only pass on all the wealth to period 2, and in period 2 the agent chooses how much to consume. Thus, in this case, payoffs are given by

$$\begin{aligned}&U_3(c_3) = u(c_3),\\ &U_2\left(c_2, c_3\right) = u(c_2) + v(c_2) - v(c_2 + c_3) + u(c_3),\\ &U_1\left(c_2, c_3\right) = U_2\left(c_2, c_3\right)\end{aligned}$$

Let c_2^U, c_3^U be the solution of this problem when no illiquid assets are available, i.e., the non commitment solution. The first order conditions for this solution are:

$$u'(c_2^U) = u'(c_3^U) - v'(c_2^U), \tag{12}$$

In contrast, when illiquid assets are available, the situation is quite different. In this case the maximal feasible amount of consumption by agent 2 is s_{12}, agent 1's saving choice. Therefore, self 1, by choosing $s_{12} < k$, can reduce the temptation of self 2 with respect to the case of illiquid assets. This will indeed be the case at equilibrium with illiquid assets.[3] Let us begin the characterization of equilibrium with period 3. Given savings s_{13} in illiquid assets in the first period as well as savings in the second period s_{23}, utility in the third period is

$$U_3 = u(s_{13} + s_{23}).$$

In period 2, given savings s_{13} in illiquid assets and s_{12} in assets that are now liquid, utility is given by

$$U_2 = u\left(s_{12} - s_{23}\right) + v\left(s_{12} - s_{23}\right) - v\left(s_{12}\right) + u(s_{13} + s_{23}).$$

As we noticed, if $s_{12} < k$, the fact that assets s_{13} are illiquid reduces the temptation for the agent in period 2. Thus, the optimal solution in period 1 is to choose s_{12}, s_{13} to maximize

$$U_1 = u\left(s_{12}\right) + u\left(s_{13}\right)$$

[3] Assuming that self 1 does not consume and hence experience no instantaneous temptation induces a more clear-cut result, but the same arguments would go through if we were to allow for period 1 consumption.

because, by ensuring that $s_{12} = c_2$, this eliminates temptations in period 2. Let c_2^*, c_3^* be the solution to this maximization problem, i.e., the commitment solution. Note that c_2^*, c_3^* satisfies:

$$u'(c_2) = u'(c_3). \tag{13}$$

Contrasting equations (13) and (12) highlights the demand for commitment. Indeed, absent commitment, in period 2, the agent would want to shift resources from period 3 to period 2 whenever $v' > 0$ and $u''(c) + v''(c) < 0$.

We now introduce the possibility of government debt. For the purpose of this Web Appendix we assume that there are no distortions in order to make the comparison with the $\beta - \delta$ model used in the paper more direct.

Assume that d^e is the candidate equilibrium level of government debt. From the optimal savings and portfolio choices of the agent we must have:

$$U_3 = u(s_{13} + s_{23} - d^e),$$
$$U_2 = u\left(s_{12} - s_{23} + d^e\right) + v\left(s_{12} - s_{23} + d^e\right) - v\left(s_{12} + d^e\right) + u(s_{13} + s_{23} - d^e).$$

So, if $d^e \leq c_2^*$, the optimal solution in period 1 sets $s_{12} = c_2^* - d^e$, $s_{13} = c_3^* + d^e$ which allows restoring the full commitment utilities in all periods.

However, as long as the debt limit $\overline{d}$ is below the non-commitment level of consumption, c_2^U, the equilibrium debt will be raised up to the debt limit. Consider on the contrary a debt level d such that $d < \overline{d} \leq c_2^U$, in period 2, the **actual** payoff function determining voting over government debt that candidates implicitly maximize is

$$U_2 = u\left(c_2^* + d\right) + v\left(c_2^* + d\right) - v\left(s_{12} + \overline{d}\right) + u(s_{13} + s_{23} - d).$$

Thus, whenever $d < \overline{d}$, the agent has an incentive to vote for higher debt.

This reasoning can easily be extended to show that when $\overline{d} > c_2^U$, then equilibrium debt is equal to c_2^U thus showing the analogue of our Proposition 1 for the case of Gul and Pesendorfer preferences. The case of distortions can also be treated in a similar fashion.

3 Allowing for Period One Consumption

The paper focused on an environment in which consumption takes place only in periods 2 and 3. In principle, individuals could also make consumption decisions while planning for future consumption. Foreseeing their future behavior, individuals can then adjust their immediate consumption and thereby affect their future budget. We now consider such settings. As in the paper, there is a measure 1 of voters who live for three periods. In period 1 voters have a wealth k from which to finance consumption over three periods. Preferences over consumption sequence c_1, c_2, c_3 are given by

$$
\begin{aligned}
&U_1(c_1, c_2, c_3) = u(c_1) + \beta\delta u(c_2) + \beta\delta^2 u(c_3), \\
&U_2(c_2, c_3) = u(c_2) + \beta\delta u(c_3), \\
&U(c_3) = u(c_3),
\end{aligned}
\tag{14}
$$

where u is a continuous and strictly concave utility function. We also assume that the utility function is three times continuously differentiable. As in the paper, we assume that $\delta = 1$ and that agents are sophisticated. We use the notation used in the paper for the commitment and no-commitment consumption choices.

While period-one consumption may affect the budget left for one's period-two self, the demand for commitment is similar to that without period-one consumption. Namely, commitment leads to lower second period consumption: $c_2^* < c_2^U$.

Consider first the benchmark in which debt is non-distortionary.

In period 1 an agent who predicts equilibrium per-capita debt levels of d, chooses savings intended for period 2, denoted by s_{12} and for period 3, denoted by s_{13}, to solve

$$\max_{s_{12}, s_{13}} u(c_1) + \beta u(s_{12} + d - s_{23}) + \beta u(s_{13} + s_{23} - d).$$

In period 2 a voter with preference parameter β chooses savings s_{23} to solve

$$\max_{s_{23}} u(s_{12} + d - s_{23}) + \beta u(s_{13} + s_{23} - d).$$

The political process proceeds as in the paper.

3.1 Equilibrium Characterization

The Incomplete Ricardian Equivalence characterized in Proposition 1 in the paper still holds. Namely, we have that:

Proposition 1 (Incomplete Ricardian Equivalence)

1. *If $\overline{d} \leq c_2^*$ then both candidates offer platforms with debt $\overline{d}$. Equilibrium consumption is (c_1^*, c_2^*, c_3^*).*

2. *If $c_2^* < \overline{d} < c_2^U$ then both candidates offer platforms with debt $\overline{d}$. In equilibrium, second-period consumption is $c_2 = \overline{d}$.*

3. *If* $\overline{d} \geq c_2^U$ *then any* d such that $c_2^U \leq d \leq k$ *is part of an equilibrium. Equilibrium consumption is* $\left(c_1^U, c_2^U, c_3^U\right)$.

Proof. 1. Assume by way of contradiction that equilibrium debt is $d^* < \overline{d}$. If this is the case, a voter can implement the commitment sequence of consumption c_1^*, c_2^*, c_3^* by choosing $s_{12} = c_2^* - d^*$, and $s_{13} = c_3^* + d^*$. This is feasible since $d^* < \overline{d} < c_2^*$. Hence, these are the optimal choices for the voter. But, by definition of c_2^*, c_3^*, $u'\left(c_2^*\right) > \beta u'\left(c_3^*\right)$, and therefore, in period 2 all voters would vote for a candidate who offered a slightly higher debt. Thus, the only debt that can be part of an equilibrium is $\overline{d}$. Given a debt of $\overline{d}$, in period 1, each voter chooses $s_{12} = c_2^* - \overline{d}$, $s_{13} = c_3^* + \overline{d}$. Given these saving choices, none of the voters would vote for a candidate that offered a lower debt in the second period, proving that debt and this sequence of consumption constitute a unique equilibrium.

2. Assume by way of contradiction that, in equilibrium, a debt $d^* < \overline{d}$ is implemented. As in part (1), voters choose savings to restore commitment as much as possible. Assume that $c_2^* < d^*$ (otherwise, the proof of part (1) applies). Each agent maximizes

$$\begin{aligned} & u\left(c_1\right) + \beta u\left(c_2\right) + \beta u\left(k - c_1 - c_2\right) \\ s.t. \quad c_2 \ \geq \ & d^*. \end{aligned}$$

The first order conditions yield

$$u'\left(c_1\right) = \beta u'\left(k - c_1 - d^*\right) > u'\left(c_2\right) = u'\left(d^*\right)$$

because $d^* > c_2^*$ (recall that $u'\left(c_2^*\right) = u'\left(c_3^*\right)$). This means that the agent sets $s_{12} = 0$ since second-period consumption is already higher than desired by the first-period self. However, since $d^* < c_2^U$, $u'(d) > \beta u'(c_3)$. Thus, in period 2 all voters would vote for higher debts contradicting the assumption that d is an equilibrium debt level. Finally, to conclude that a debt of $\overline{d}$ is indeed part of an equilibrium, observe that, given $\overline{d}$, by similar reasoning, the optimal saving choices of all voters would lead to $u'\left(\overline{d}\right) > \beta u'(c_3)$. Thus, no voter would vote for lower debts.

3. We first show that the claimed outcomes are part of an equilibrium. Given any candidate equilibrium debt $k > d^* \geq c_2^U$ that is expected by voters in period 1, an optimal policy of a voter in period 1 is a choice of $s_{12} = 0$ and $s_{13} = c_3^U - \left(d^* - c_2^U\right)$. In addition, given d^*, in equilibrium, $s_{23} = d^* - c_2^U$ is to be saved in period 2 for period 3. Given this policy, by the definition of c_2^U, c_3^U, we have

$$u'\left(c_2^U\right) = \beta u'\left(c_3^U\right)$$

giving no incentive to any period-2 self to change her savings plan away from s_{23}. Suppose now that the period-1 self were to change (e.g., increase) s_{13}. Then, the period-2 self would make an offsetting change (reduction) in s_{23} to restore period 2 optimality. Any change in s_{12} would similarly be offset (recall that since $d^* \geq c_2^U$, even if $s_{12} = 0$, the period-2 self can unilaterally choose c_2^U). Thus, the period-1 self has no incentive to deviate.[4]

Given these policies for the voters, consider a deviation to $d < d^*$ in period 2. As long as the deviation is small ($d \geq c_2^U$), all voters are indifferent (they can just make an offsetting reduction in s_{23} to restore the desired consumption sequence). If the deviation is large ($d < c_2^U$), then voters who can no longer make such offsetting reduction in s_{23}. All voters would therefore vote against a candidate offering such a deviation. A deviation to $d > d^*$ would leave all voters indifferent because they could make offsetting changes in s_{23}.

Consider now a candidate equilibrium debt $d^* < c_2^U$. Such an expected debt would constrain period-2 consumption for the voters, leading to victory in period 2 for a candidate offering $d > d^*$. ■

When debt is distortionary, the analysis changes slightly when one accounts for consumption in the first period. The equilibrium characterization is analogous to that corresponding to the case in which consumption occurs only in the second and third periods. Indeed, let $c_1^*(d), c_2^*(d)$, and $c_3^*(d)$ be the commitment sequence of consumption given debt d, namely, the solution to the following problem:

$$\begin{aligned} & \max \{u(c_1) + \beta(u(c_2) + u(c_3))\} \\ s.t. \quad c_1 + c_2 + c_3 &= k - \eta d \end{aligned}$$

Analogously, let $c_1^U(d), c_2^U(d)$, and $c_3^U(d)$ be the corresponding quantities without commitment. We define d^* as the solution of $c_2^*(d^*) = d^*$.[5]

We now introduce an artificial constrained-maximization problem for a voter of preference parameter $\beta(1+\eta) < 1$.

$$\begin{aligned} & \max u(c_1) + \beta[u(c_2) + u(c_3)] \\ s.t. \quad u'(c_2) &= \beta(1+\eta) u'(c_3), \\ c_1 + c_2 + c_3 &= k - d\eta. \end{aligned} \tag{15}$$

[4] There are multiple ways for the period-1 self to implement the uncommitted sequence, involving increasing s_{12} and s_{23} by the same amounts with offsetting reductions to s_{13}. All these are weakly dominated by the proposed sequence.

[5] Notice that $c_2^*(0) \geq 0$, while $c_2^*(k/\eta) = 0 < k/\eta$, and so the Intermediate Value Theorem guarantees the existence of such a d^*.

Notice that when there is consumption in the first period, the optimal consumption is not simply prescribed by the second-period constraint, since the resources available to the second-period self are endogenous and determined by consumption in the first period. Denote by $(c_1^\eta(d), c_2^\eta(d), c_3^\eta(d))$ the consumption sequence that solves the problem . We now define d^{**} to be the solution of $d^{**} = c_2^\eta(d^{**})$.[6] It is easy to show that $d^* < d^{**}$.

Proposition 2 (Distortionary Equilibrium Debt)

1. *If* $\beta(1+\eta) > 1$ *then in equilibrium there is no debt and consumption is given by* (c_1^*, c_2^*, c_3^*).

2. *Assume that* $\beta(1+\eta) < 1$. *If* $\overline{d} \leq d^*$, *then equilibrium debt is given by* $\overline{d}$ *and consumption is given by* $\left(c_1^*\left(\overline{d}\right), c_2^*\left(\overline{d}\right), c_3^*\left(\overline{d}\right)\right)$. *If* $d^* < \overline{d} \leq d^{**}$, *then equilibrium debt is given by* $\overline{d}$ *and period* 2 *consumption is given by* $c_2 = \overline{d}$. *If* $\overline{d} > d^{**}$, *then debt is given by* d^{**} *and period* 2 *consumption is given by* $c_2 = d^{**}$.

Proof. 1. We first show that there is an equilibrium with zero debt. Given an expected second-period debt of zero, in period 1 voters choose the mix of liquid and illiquid assets $s_{12} = c_2^*$ and $s_{13} = c_3^*$ that implements the commitment consumption sequence (c_1^*, c_2^*, c_3^*). Given this mix of savings, $u'(c_2^*) = u'(c_3^*)$. Thus, if $\beta(1+\eta) > 1$, $u'(c_2^*) < \beta(1+\eta)u'(c_3^*)$ and voters have no incentive to vote for positive debt. Consider now any level of expected debt d. The mix of savings has to be such that $u'(s_{12}+d) \leq u'(s_{13}+s_{23}-d)$. But then $u'(s_{12}+d) < \beta(1+\eta)u'(s_{13}+s_{23}-d)$, inducing voters to vote to reduce debt.

2. Consider now the case in which $\beta(1+\eta) < 1$. Given any $\overline{d} < d^*$ and any expected $d \leq \overline{d}$, optimal savings in period 2 are given by $s_{23} = 0$ and s_{12}, s_{13} are such that $u'(s_{12}+d) = u'(s_{13}-d)$. Thus, $u'(s_{12}+d) > \beta(1+\eta)u'(s_{13}-d)$ and voters would vote to increase debt. Thus, in this scenario equilibrium debt must be $\overline{d}$ and consumption must be given by $\left(c_1^*\left(\overline{d}\right), c_2^*\left(\overline{d}\right), c_3^*\left(\overline{d}\right)\right)$. If $d^* < \overline{d} \leq d^{**}$, then, by the same reasoning, equilibrium debt must be at least d^*. But then, by the definition of d^*, debt is higher than second-period commitment consumption, and optimal savings are at a corner: $s_{12} = s_{23} = 0$, implying that $c_2 = d$. Because $d < d^{**}$, we then have that $\beta(1+\eta)u'(c_3) < u'(c_2) < u\prime(c_3)$. This implies that voters vote for higher debt unless $d = \overline{d}$. Finally, If $\overline{d} \geq d > d^{**}$, then by the definition of d^{**}, $u'(d) < \beta(1+\eta)u'(c_3)$, so voters would vote to reduce debt. This proves that, for any $\overline{d} \geq d^{**}$ equilibrium debt is given by d^{**}. ■

[6] Again, the Intermediate Value Theorem assures that such d^{**} always exists since $c_2^\eta(0) = c_2^U(0) \geq 0$, and $c_2^\eta(k/\eta) = 0 < k/\eta$, and the Theorem of the Maximum implies that $c_2^\eta(d)$ is continuous.

3.2 Welfare Analysis

When consumption takes place only in periods 2 and 3, the analysis of the impact of distortions on welfare is dramatically simplified. Indeed, equilibrium consumption is essentially governed by the second-period constraint. Technically, we can use the implicit function theorem to derive a full ranking of welfare for different distortion levels η. When consumption occurs in period 1 as well, the budget available in period 2 is endogenous and may depend on η. Nonetheless, we can still determine the detrimental effects of distortions, as well as the impacts of suffering from self-control problems. The following result provides a comparison of equilibrium welfare with and without distortions when debt limits are large (namely, $\overline{d} > d^{**}$).

Proposition 3 (Welfare Effects of Distortions) *Whenever $\beta < \beta(1+\eta) < 1$ the equilibrium with distortions determined by η leads to lower first period welfare than the equilibrium corresponding to no distortions, when $\eta = 0$. If $\beta(1+\eta) > 1$, then first period welfare is higher than that induced by any $\beta(1+\eta) < 1$.*

Proof. Consider the following maximization problem:

$$\begin{aligned} &\max u(c_1) + \beta\left[u(c_2) + u(c_3)\right] \\ &s.t.\ \ u'(c_2) = \beta(1+\eta)u'(c_3) \\ &c_1 + c_2 + c_3 = k - \eta c_2. \end{aligned} \tag{16}$$

This is an artificial problem corresponding to an agent who chooses the debt level and her consumption plan in tandem but consuming c_2 destroys resources just as debt does. In particular, this problem generates a higher overall utility (from period 1's perspective) than that experienced by an agent who consumes $c_1^{\eta}(d^{**}), c_2^{\eta}(d^{**}), c_3^{\eta}(d^{**})$ because such an agent takes the equilibrium level of debt as given and cannot alter it unilaterally. The latter generates the equilibrium level of welfare for distortions η. Furthermore, the two coincide when $\eta = 0$. We now show that the maximized objective of problem (16) is decreasing in η. Indeed, suppose $\eta_1 > \eta_2$. Denote the solution of (16) for distortions η_1 by (c_1, c_2, c_3). We now approximate a policy under distortions η_2 small enough that it satisfies the constraints and generates a strictly higher value for the objective.

For η_2 close enough to η_1, there exists $\varepsilon > 0, \varepsilon < c_3$ such that

$$u'(c_2) = \beta(1+\eta_2)u'(c_3 - \varepsilon).$$

Therefore,

$$u'(c_2) = \beta(1+\eta_2)\left[u'(c_3) - \varepsilon u''(c_3) + O(\varepsilon^2)\right].$$

Since (c_1, c_2, c_3) is a solution to the problem with distortions η_1, $u'(c_2) = \beta(1 + \eta_1)u'(c_3)$. It follows that:

$$\varepsilon = \frac{(\eta_2 - \eta_1)\, u'(c_2)}{\beta(1 + \eta_2)u''(c_3)} + O(\varepsilon^2).$$

Consider then the policy $(c_1 + \varepsilon + (\eta_1 - \eta_2)\, c_2, c_2, c_3 - \varepsilon)$ when the distortions are η_2. Notice that, by construction, this policy satisfies the two constraints in problem (16). The difference between the generated objective and the maximal value of the objective under distortions η_1 is then:

$$\Delta = [u(c_1 + \varepsilon + (\eta_1 - \eta_2)\, c_2) - u(c_1)] + \beta\, [u(c_3 - \varepsilon) - u(c_3)]\,.$$

Using a first order approximation,

$$\begin{aligned}
\Delta &= (\varepsilon + (\eta_1 - \eta_2)\, c_2)\, u'(c_1) - \beta\varepsilon u'(c_3) = \\
&= (\eta_1 - \eta_2)\, c_2 u'(c_1) + \frac{(\eta_2 - \eta_1)\, u'(c_2)u'(c_1)}{\beta(1 + \eta_2)u''(c_3)} - \frac{(\eta_2 - \eta_1)\, u'(c_2)u'(c_3)}{(1 + \eta_2)u''(c_3)} + O(\varepsilon^2) \\
&= \frac{(\eta_1 - \eta_2)}{(1 + \eta_2)} u'(c_2) \left[\frac{u'(c_1)c_2}{u'(c_2)} - \frac{u'(c_1) - \beta u'(c_3)}{\beta u''(c_3)} \right] + O(\varepsilon^2).
\end{aligned}$$

Notice that the solution to problem (16) with distortions η_1 must satisfy $u'(c_1) = \beta\, [u'(c_2) + u'(c_3)]$ and so:

$$\Delta = \frac{(\eta_1 - \eta_2)}{(1 + \eta_2)} u'(c_2) \left[\frac{u'(c_1)c_2}{u'(c_2)} - \frac{u'(c_2)}{u''(c_3)} \right] + O(\varepsilon^2),$$

which from concavity of the instantaneous utility u, is positive whenever η_1 and η_2 are close enough. In particular, the optimal solution for problem (16) with distortions η_2 must generate a strictly higher level of the objective function than the solution with distortions η_1. It follows that welfare in our distortion economy is lower under any $\eta > 0$ relative to the case of $\eta = 0$.

Last, notice that when $\beta(1 + \eta) < 1$, all agents achieve their commitment solution absent debt, an consequently the maximal period 1 utility under the budget constraint. From Proposition 2, this is no longer the case when $\beta(1 + \eta) > 1$ and so period 1 utility is lower for distortions exceeding $1 - \beta$. ■

As in the model analyzed in paper, there are two contrasting effects of positive distortions. On the negative side, given that there is debt in equilibrium, the presence of distortions causes wealth destruction. On the positive side, distortions relax the commitment constraint in the artificial maximization that determines equilibrium debt. In fact, when η is very high ($\eta > 1 - \beta$), distortions serve as a full commitment device since, in equilibrium, voters do not vote for positive debt in the second period. The proposition shows that the negative effect dominates.

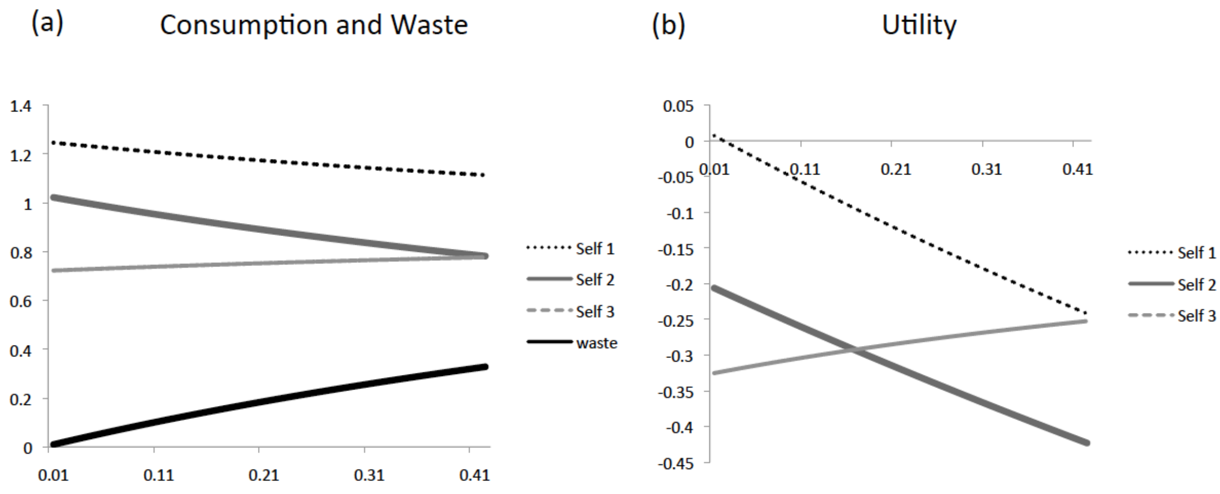

Figure 1: Outcomes for Log Instantaneous Utility ($k = 3$, $\beta = 0.7$)

Figure 1 illustrates the impact of distortions in the case of instantaneous log-utility, where we take the budget to be $k = 3$ and the population time preferences to be $\beta = 0.7$. The left panel of the figure illustrates the consumption patterns and wealth destroyed. Notice that consumption declines with η in periods 1 and 2, but is increasing in period 3. This reflects the two effects discussed above that distortions have – on the one hand, they destroy wealth, and indeed, wealth destruction increases with η; On the other hand, they relax the constraints in period 2, which allows for more delayed consumption. The right panel of the figure illustrates the impact of distortions on welfare from the perspective of each self. Welfare for period-1 and period-2 selves declines with η, in line with the statement in the proposition. This indicates that the effect of wealth destruction outweighs the benefits of smoothing derived from greater distortions, and so overall greater distortions do not help individuals early in the process. However, since period 3 consumption is increasing, so does welfare in period 3.

3.3 Heterogeneity

We now consider what happens when agents are heterogeneous in their present-bias parameter β. In analogy to our previous notation, we will denote by $c_t^*(\beta; d)$ and $c_t^\eta(\beta; d)$ the commitment solution for debt d and the solution to the constrained problem (15) for each individual of preference parameter β.

We start by assuming that second period consumption $c_2^\eta(\beta; d)$ increases monotonically in β. This holds when the utility function has sufficient curvature. We note that there are

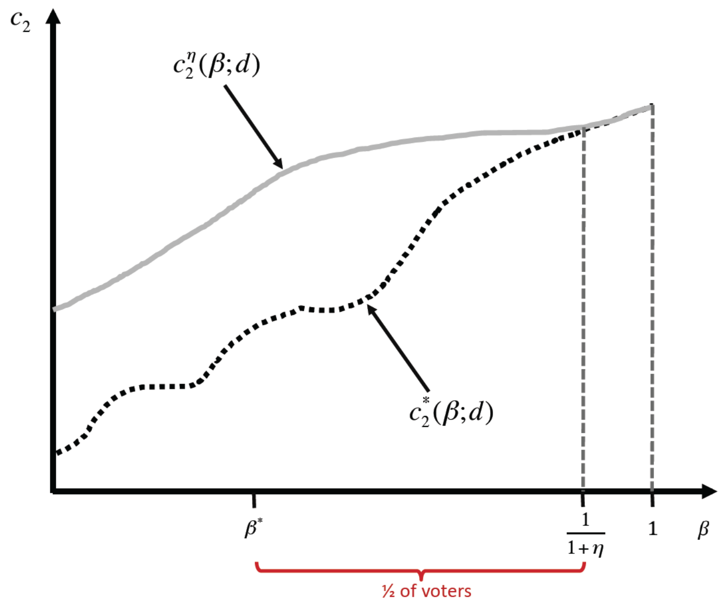

Figure 2: Consumption Patterns for a Given Debt Level

many preferences for which this does not hold. For instance, with log utility, consumption is not monotonic. However, even in such a case our initial discussion will be valid for a fairly wide class of distributions of the β parameter. We discuss the more general case below. We note that this assumption stands in stark contrast with the environment in which there is no consumption in the first period. Indeed, in that case $c_2^{\eta}(\beta; d)$ is decreasing and $c_t^{*}(\beta; d)$ is a constant function independent of β.

Let β^* be such that $G(\frac{1}{1+\eta}) - (\beta^*) = 1/2$. That is, half the population has preferences that are between β^* and $\frac{1}{1+\eta}$. Figure 3 depicts the shape of commitment and no-commitment consumption levels in period 2 as a function of preferences for a particular debt level.

The agent of type β^* turns out to be the pivotal agent for determining debt in this environment. We can now define $d^*(\beta^*)$ and $d^{**}(\beta^*)$ as the solutions of $d^* = c_2^*(\beta^*, d^*)$ and $d^{**} = c_2^{\eta}(\beta^*, d^{**})$.[7]

1. *If* $\beta_M(1+\eta) > 1$, *then in equilibrium there is no debt, and consumption is given by* $c_1^*(\beta), c_2^*(\beta), c_3^*(\beta)$.

2. *Assume that* $\beta_M(1+\eta) < 1$. *If* $\overline{d} \leq d^{**}(\beta^*)$, *then equilibrium debt is given by* $\overline{d}$. *If* $\overline{d} > d^{**}(\beta^*)$, *then debt is given by* $d^{**}(\beta^*)$.

[7] Existence and uniqueness of these debt levels follow the same arguments used for the case of a homogenous electorate.

3. *For any equilibrium debt level d, individual consumption for an agent of preference parameter β, period-2 consumption level in equilibrium is given by:*

$$c_2(\beta; d) = \begin{cases} c_2^{\eta}(\beta; d) & \beta \leq \beta_L(d) \\ d & \beta_L(d) \leq \beta < \beta_H(d) \\ c_2^{*}(\beta; d) & \beta \geq \beta_H(d) \end{cases}.$$

With respect to the distribution of preferences, notice that a shift in distribution changes the debt structure in the economy only when it modifies the preferences β^* of the 'pivotal agent'. As β^* increases, $c_2^*(\beta^*; d)$ and $c_2^{\eta}(\beta^*; d)$ increase for all d, and therefore both d^* and d^{**} increase.

We say G' is a *median preserving spread* of G if both share the same median β_M and for any $\beta < \beta_M$, $G'(\beta) \geq G(\beta)$, while for any $\beta > \beta_M$, $G'(\beta) \leq G(\beta)$. Intuitively, this implies that, under G', more weight is put on more extreme values of β (see Malamud and Trojani (2009) for applications to a variety of other economic phenomena).

The above discussion then implies the following corollary.

Corollary 2 (Distributional Shifts)

1. *Assume $G(\frac{1}{1+\eta}) = G'(\frac{1}{1+\eta})$. If G' First Order Stochastically Dominates G, and the corresponding medians $\beta_M, \beta'_M < \frac{1}{1+\eta}$, then equilibrium debt under G' is (weakly) higher than that under G.*

2. *If G' is a Median Preserving Spread of G, then equilibrium debt under G' is (weakly) lower than that under G.*

Part 1 of this corollary says that, as the population becomes more "virtuous" or less subject to self-control problems, equilibrium debt increases. This is potentially surprising but is a natural consequence of the logic of our model. There are two ways to glean intuition for this result. The more mechanical one is to recall that equilibrium debt is equal to second period consumption. As β^* increases, so does the desired second period consumption of the pivotal agent β^*. Thus, equilibrium debt increases. Alternatively, notice that in our model debt arises because of the desire of the pivotal agent to constrain her future self, and the subsequent response of the political system undoing this commitment. The more virtuous the pivotal agent, the higher the level of debt that is required to prevent this agent from attempting to commit at an even higher level.

We now discuss the more general case in which second period consumption may not be increasing in β. For any η, denote by d^p the debt level such that:

$$G\left\{\beta \mid c_2^\eta(\beta; d^p) < d^p\right\} = \frac{1}{2}.$$

Proposition 6 can now be restated with d^p playing the role of $d^{**}(\beta^*)$. If second period consumption is *decreasing* in β, then d^p will correspond to $c_2^\eta(\beta_M; d^{**})$: the median voter will be pivotal. Otherwise, there may be multiple pivotal voters.

We now discuss how the welfare of different agent types is affected by the presence of illiquid assets. Our result in Proposition 5 showing that agents would be made better off in the first period if illiquid assets were penalized obviously extends to the case where the degree of heterogeneity is limited. Furthermore, if $c_2^\eta(\beta, d)$ is increasing in β, it is possible to show that, for any degree of heterogeneity, all agents with $\beta \leq \beta^*$ as well as those with sufficiently high β are made worse off by the presence of illiquid assets: the former group because for these types, debt is higher than $c_2^\eta(\beta; d)$ and second period consumption is completely out of transfers, so the logic of Proposition 5 immediately holds for these agents; the latter group because these types do not have much of a self-control problem, so the presence of illiquid assets gains them little commitment but generates a destruction of resources through debt.

References

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