D Addendum: General Case of CAPM Economy and Its Competitive Equilibrium

The economy is populated by H agents, F firms. Agent h's preferences are represented by a Constant Absolute Risk Aversion utility function $u^h(\cdot)$ over the consumptions at date 0 and date 1, denoted as c_0^h and c_1^h, respectively:

$$u^h(c_0^h, c_1^h) \equiv -\frac{1}{A} e^{-Ac_0^h} - \frac{1}{A} e^{-Ac_1^h} \ . \tag{D.1}$$

The economy has a N-dimensional orthogonal normal factor structure $(x_1, \ldots, x_n)$, which is a multivariate normal with mean 0, and variance–covariance matrix (normalized to) I, the identity matrix. In particular, each agent h's endowments in period 1, y_1^h, is generated as a linear combination of N underlying normal risk factors, and hence is in general correlated with other agents' endowments:

$$y_1^h - E(y_1^h) \equiv \sum_{n=1}^{N} \beta_n^h x_n \ , \ h = F+1, ..., H. \tag{D.2}$$

The first $F < H$ agents are the entrepreneurs. Entrepreneur h owns the firm $f(= h)$.

Each firm's cash flow, y_1^f, is also generated by the N factors. Without loss of generality, we assume that the stock market risk is driven by $C < N$ common orthogonal factors, $(x_1, ..., x_C)$, and F orthogonal factors, $(x_{C+1}, ..., x_{C+F})$, which correspond to the sectoral risk added by each firm's cash flow:

$$y_1^f - E(y_1^f) \equiv \sum_{c=1}^{C} \beta_c^f x_c + \ \beta_f^f x_{C+f}, \ f = 1, ..., F. \tag{D.3}$$

Purely financial assets have payoff z_i, $i = 1, ..., I$, which in terms of the factor structure is written as

$$z_i - E(z_i) \equiv \sum_{c=1}^{C} \beta_c^i x_c + \sum_{i=1}^{I} \beta_j^i x_{C+F+i}, \ i = 1, ..., I, \tag{D.4}$$

where $(x_{C+F+1}, ..., x_{C+F+I})$ contains the additional risks in the return structure of financial markets.

In our economy, all agents can trade the I financial assets and the C orthogonal factors. Moreover, there exists a stock market for trade of the F firms, although the participation of entrepreneurs (agents $h \leq F$) is restricted; in particular, entrepreneur f cannot trade his own firm f, $f = 1, \ldots, F$. Using the I financial assets, the C markets for factors, and the market for the F stocks, under the assumptions of payoff orthogonality, agents can replicate the payoffs of first $C + F + I$ risk factors. (Participation restrictions in the stock market translate into analogous participation restrictions in the market for the F firms' risk factors. Similarly, the positive net supply of stocks translates into positive net supply of all factors.) Therefore it is formally equivalent, and it turns out to be convenient to represent the competitive equilibrium of our CAPM economy in terms of not only the market but also the prices of the $C + F + I$ risk factors. We do this in the following.

We use a single index for all factors: $j \in \mathcal{J} \equiv \{1, \ldots, J\}$, where $J \equiv C + F + I$. In general, $J < N$, and the set of risky assets traded by agent h, denoted as $\mathcal{J}^h$, may be a proper subset of $\mathcal{J}$, that is, $\mathcal{J}^h \subset \mathcal{J}$ for some h, but we assume that all agents h are allowed to trade the risk-free bond. H_j denotes the set of agents trading asset j, $|H_j|$ being its size.

The problem of each agent h is to choose a consumption allocation at time 0, c_0^h, portfolio positions in the risk-free bond and in all tradable assets, $[\theta_0^h, \theta_j^h]_{j \in \mathcal{J}}$, and a consumption allocation at time 1, a random variable c_1^h, to maximize the expected utility

$$E\left[u^h(c_0^h, c_1^h)\right] \equiv -\frac{1}{A} e^{-Ac_0^h} + E\left[-\frac{1}{A} e^{-Ac_1^h}\right], \tag{D.5}$$

subject to the budget constraints and the restricted participation constraints:

$$\begin{aligned}
c_0^h &= y_0^h - \pi_0 \theta_0^h - \sum_{j \in \mathcal{J}} \pi_j \theta_j^h, \ h > F && \text{(D.6)}\\
c_0^h &= y_0^h + w^h p^h - \pi_0 \theta_0^h - \sum_{j \in \mathcal{J}} \pi_j \theta_j^h, \ h \in H, \ h \leq F && \text{(D.7)}\\
c_1^h &= y_1^h + \theta_0^h + \sum_{j \in \mathcal{J}} \theta_j^h x_j, \ h > F && \text{(D.8)}\\
c_1^h &= (1 - w^h) y_1^h + \theta_0^h + \sum_{j \in \mathcal{J}} \theta_j^h x_j, \ h \leq F && \text{(D.9)}\\
\theta_j^h &= 0, \ j \notin \mathcal{J}^h. && \text{(D.10)}
\end{aligned}$$

Note that the budget constraint for entrepreneur h includes the time-0 proceeds from the sale of a fraction w^h of his firm amounting to $w^h p^h$. As discussed in the paper, under rational expectations, the price of the firm p^h is given by $p^h = \pi_0 E(y_1^h) + \sum_{1 \leq j \leq J} \pi_j \beta_j^h$. Let s_j^h denote the positive supply of risk factor j provided by the entrepreneur h through the sale of fraction w^h of his firm. Under the factor decomposition (equation D.3) for each firm's cash flows, these positive supplies are given by $s_0^h = w^h E(y_1^h)$ and $s_j^h = w^h \beta_j^h$, $1 \leq j \leq J$, so that the proceeds from sale of the firm, $w^h p^h$, can also be expressed as $w^h p^h = \sum_{0 \leq j \leq J} \pi_j s_j^h$.

Definition D.1 *A competitive equilibrium is a consumption allocation* (c_0^h, c_1^h)*, for all agents* $h \in \mathcal{H}$*, which solves the problem of maximizing (D.5) subject to (D.6–D.10) at prices* $\pi \equiv [\pi_0, \pi_j]_{j\in\mathcal{J}}$*, and such that consumption and financial markets clear*

$$\sum_h \left(c_0^h - y_0^h\right) \leq 0, \tag{D.11}$$

$$\sum_h \left(c_1^h - y_1^h\right) \leq 0, \text{ with probability } 1 \text{ over } \Omega, \text{ and} \tag{D.12}$$

$$\sum_h \theta_j^h = s_j, \; j = 0, 1, \ldots, J, \tag{D.13}$$

where s_j is the net supply of factor j, $s_j \equiv \sum_{1\leq h\leq F} s_j^h$.

Proposition D.2 *The competitive equilibrium of the two-period CAPM economy, defined by equations (D.5)–(D.10), with the market-clearing condition given by equations (D.11)–(D.13), is characterized by prices of assets* (π_j)*, portfolio choices* (θ_j^h)*, and consumption allocations* (c_t^h)*, given below.*

$$\pi_0 = exp\left\{A\left(y_0 - Ey_1\right) + \frac{A^2}{2H}\sum_{h=1}^{H}\left[(1-R_h^2)var(y_1^h) + \sum_{j\in J^h}\left(\beta_j + \frac{1}{|H_j|}s_j\right)^2\right]\right\}, \tag{D.14}$$

where

$$y_0 = \frac{1}{H}\sum_{h=1}^{H} y_0^h, \quad y_1 = \frac{1}{H}\sum_{h=1}^{H} y_1^h, \tag{D.15}$$

$$\beta_j = cov\left[\frac{1}{|H_j|}\left(\sum_{h\in H_j, h\leq F}(1-w^h)y_1^h + \sum_{h\in H_j, h>F} y_1^h\right), x_j\right], \tag{D.16}$$

$$s_0^h = w^h E(y_1^h), \quad s_j^h = w^h \beta_j^h, \; 1 \leq j \leq J, \quad s_j = \sum_{1\leq h\leq F} s_j^h, \; 0 \leq j \leq J, \tag{D.17}$$

$$\frac{\pi_j}{\pi_0} = E\left(x_j\right) - A\left(\beta_j + \frac{1}{|H_j|}s_j\right), \tag{D.18}$$

and for $h > F$ (non-entrepreneurs),

$$R_h^2 \equiv \frac{\sum_{j\in J^h} \left(\beta_j^h\right)^2}{var(y_1^h)} \ , \tag{D.19}$$

$$\theta_j^h = \left(\beta_j + \frac{1}{|H_j|} s_j\right) - \beta_j^h, \ \ j \in \mathcal{J}^h, \quad and\ \theta_j^h = 0, \ \ j \in (\mathcal{J}^h)^c, \tag{D.20}$$

$$\theta_0^h = \frac{1}{1+\pi_0}\left(y_0^h - E(y_1^h) - \sum_{j\in\mathcal{J}^h} \pi_j \theta_j^h + \frac{A}{2}\ var(c_1^h) - \frac{1}{A}\ ln(\pi_0)\right), \tag{D.21}$$

$$c_1^h = \theta_0^h + \sum_{j\in\mathcal{J}^h}\left(\beta_j + \frac{1}{|H_j|} s_j\right) x_j + \left(y_1^h - \sum_{j\in\mathcal{J}^h} \beta_j^h x_j\right), \tag{D.22}$$

$$var(c_1^h) = var(y_1^h) - \sum_{j\in\mathcal{J}^h} (\beta_j^h)^2 + \sum_{j\in\mathcal{J}^h}\left(\beta_j + \frac{1}{|H_j|} s_j\right)^2, \tag{D.23}$$

$$c_0^h = -\frac{1}{A}\ ln\ \frac{1}{\pi_0} + E(y_1^h) + \theta_0^h - \frac{A}{2}\ var(c_1^h), \tag{D.24}$$

and finally, for $h \leq F$ (entrepreneurs),

$$R_h^2 \equiv \frac{\sum_{j\in J^h} (1-w^h)^2 \left(\beta_j^h\right)^2}{var(y_1^h)} \ , \tag{D.25}$$

$$\theta_j^h = \left(\beta_j + \frac{1}{|H_j|} s_j\right) - (1-w^h)\beta_j^h, \ \ j \in \mathcal{J}^h, \quad and\ \theta_j^h = 0, \ \ j \in (\mathcal{J}^h)^c, \tag{D.26}$$

$$\theta_0^h = \frac{1}{1+\pi_0}\left(y_0^h + w^h p^h - (1-w^h)E(y_1^h) - \sum_{j\in\mathcal{J}^h} \pi_j \theta_j^h + \frac{A}{2}\ var(c_1^h) - \frac{1}{A}\ ln(\pi_0)\right), \tag{D.27}$$

$$c_1^h = \theta_0^h + \sum_{j\in\mathcal{J}^h}\left(\beta_j + \frac{1}{|H_j|} s_j\right) x_j + (1-w^h)\left(y_1^h - \sum_{j\in\mathcal{J}^h} \beta_j^h x_j\right), \tag{D.28}$$

$$var(c_1^h) = (1-w^h)^2 var(y_1^h) - \sum_{j\in\mathcal{J}^h} (1-w^h)^2(\beta_j^h)^2 + \sum_{j\in\mathcal{J}^h}\left(\beta_j + \frac{1}{|H_j|} s_j\right)^2, \tag{D.29}$$

$$c_0^h = -\frac{1}{A}\ ln\ \frac{1}{\pi_0} + (1-w^h)E(y_1^h) + \theta_0^h - \frac{A}{2}\ var(c_1^h). \tag{D.30}$$

This equilibrium, which exhibits a positive supply of assets, is similar to the one without positive supply (see Willen, 1997, and Acharya and Bisin, 2000), but all expressions for the entrepreneurs are modified to reflect the facts that (i) entrepreneur h holds only a fraction $(1-w^h)$ of his firm; (ii) at time 0, entrepreneur h collects proceeds for the remaining fraction w^h of his firm amounting to $w^h p^h$; and (iii) aggregate beta β_j in the case of zero-supply assets is replaced by $(\beta_j + \frac{1}{H_j} s_j)$ to reflect the positive supply of assets.

Proof: Consider the competitive equilibrium of Definition D.1. To determine the equilibrium in closed-form, we derive the first-order conditions for each agent's maximization of utility function and then apply the market-clearing conditions. Note that fractions of firms to be sold have already been determined and hence positive supplies of all assets are taken as given by all agents. Since competitive entrepreneurs cannot affect the prices of bond and risk factors (or their aggregate supplies), it follows that the proceeds collected from sales of firms are also taken as given by the respective entrepreneurs. Finally, the technology choice of each firm – the firm's cash flow betas – are also taken as given by all agents: either the betas are observed and contracted upon, as in the case of owner-managed firms with no moral hazard, or these are unobserved but rationally anticipated, as in the case of owner-managed firms with moral hazard and in the case of corporations.

The maximization problem of agent h in equation (D.5) can be cast in terms of the agent's choice of portfolios, $[\theta_0^h, \theta_j^h]_{j \in \mathcal{J}} \in \Re^{J+1}$, as

$$\max_{[\theta_0^h, \theta_j^h]_{j\in\mathcal{J}}} \; -\frac{1}{A} e^{-Ac_0^h} + E\left[-\frac{1}{A} e^{-Ac_1^h}\right], \tag{D.31}$$

subject to the constraints (D.6)–(D.10). Since all endowments and risky asset payoffs are normally distributed, this objective simplifies to

$$\max_{[\theta_0^h, \theta_j^h]_{j\in\mathcal{J}}} \; -\frac{1}{A} e^{-Ac_0^h} - \frac{1}{A} e^{-AE(c_1^h) + \frac{A^2}{2}\mathrm{var}(c_1^h)} \; . \tag{D.32}$$

Using equations (D.8)–(D.9) and the normalizations $E(x_j) = 0$, $\mathrm{var}(x_j) = 1$, $\forall j \in \mathcal{J}$, we obtain

$$E(c_1^h) = E(y_1^h) + \theta_0^h, \; h > F \tag{D.33}$$

$$E(c_1^h) = (1-w^h)E(y_1^h) + \theta_0^h, \; h \le F \tag{D.34}$$

$$\mathrm{var}(c_1^h) = \mathrm{var}(y_1^h) + \sum_{j\in\mathcal{J}^h} (\theta_j^h)^2 + 2\sum_{j\in\mathcal{J}^h} \theta_j^h \mathrm{cov}(y_1^h, x_j), \; h > F \tag{D.35}$$

$$\mathrm{var}(c_1^h) = (1-w^h)^2\mathrm{var}(y_1^h) + \sum_{j\in\mathcal{J}^h} (\theta_j^h)^2 + 2(1-w^h)\sum_{j\in\mathcal{J}^h} \theta_j^h \mathrm{cov}(y_1^h, x_j), \; h \le F. \tag{D.36}$$

Taking the first-order condition with respect to θ_0^h, we get

$$\pi_0 e^{-Ac_0^h} \;\; = \;\; E\left[e^{-Ac_1^h}\right], \; \forall h. \tag{D.37}$$

Taking the first-order condition with respect to $\theta_j^h \in \mathcal{J}^h$, we get

$$\pi_j e^{-Ac_0^h} \;\; = \;\; -A\, E\left[e^{-Ac_1^h}\right]\left(\theta_j^h + \text{cov}(y_1^h, x_j)\right), \; h > F \tag{D.38}$$

$$\pi_j e^{-Ac_0^h} \;\; = \;\; -A\, E\left[e^{-Ac_1^h}\right]\left(\theta_j^h + (1-w^h)\text{cov}(y_1^h, x_j)\right), \; h \leq F. \tag{D.39}$$

Dividing equation (D.37) by equation (D.38) for $h > F$, and by equation (D.39) for $h \leq F$, and summing up for $h \in H_j$, we obtain

$$|H_j| \, \frac{\pi_j}{\pi_0} \;\; = \;\; -A \sum_{h\in H_j} \theta_j^h - A \,\text{cov}\left(\sum_{h\in H_j, h\leq F} (1-w^h) y_1^h + \sum_{h \in H_j, h>F} y_1^h, x_j\right). \tag{D.40}$$

Dividing throughout by H_j, using the market-clearing condition (D.13), and substituting for β_j from the definition (D.16), yields the CAPM pricing relationship of (D.18):

$$\frac{\pi_j}{\pi_0} = -A\left(\beta_j + \frac{1}{|H_j|} s_j\right). \tag{D.41}$$

Substituting equations (D.37) and (D.41) in equations (D.38) and (D.39) yields the following portfolio choice θ_j^h:

$$\theta_j^h = \left(\beta_j + \frac{1}{|H_j|} s_j\right) - \beta_j^h, \;\; j \in \mathcal{J}^h, \; h > F \tag{D.42}$$

$$\theta_j^h = \left(\beta_j + \frac{1}{|H_j|} s_j\right) - (1-w^h)\beta_j^h, \;\; j \in \mathcal{J}^h, \; h \leq F, \tag{D.43}$$

where we have used the definition $\beta_1^h = \text{cov}(y_1^h, x_j)$.

In order to obtain the portfolio choice θ_0^h, we rewrite the first-order condition (D.37) as

$$\pi_0 e^{-Ac_0^h} \;\; = \;\; e^{-AE(c_1^h) + \frac{A^2}{2}\text{var}(c_1^h)}, \; \forall h. \tag{D.44}$$

Taking the natural log, substituting equations (D.6) and (D.33) for $h > F$, or equations (D.7) and (D.34) for $h \leq F$, and rearranging yields

$$\theta_0^h = \frac{1}{1+\pi_0}\left(y_0^h - E(y_1^h) - \sum_{j \in \mathcal{J}^h} \pi_j \theta_j^h + \frac{A}{2}\operatorname{var}(c_1^h) - \frac{1}{A}\ln(\pi_0)\right), \ h > F \tag{D.45}$$

$$\theta_0^h = \frac{1}{1+\pi_0}\left(y_0^h + w^h p^h - (1-w^h)E(y_1^h) - \sum_{j \in \mathcal{J}^h} \pi_j \theta_j^h + \frac{A}{2}\operatorname{var}(c_1^h) - \frac{1}{A}\ln(\pi_0)\right), \quad h \leq F. \tag{D.46}$$

Next, substituting equation (D.42) in equation (D.8) and equation (D.43) in equation (D.9) and rearranging, we obtain the three-fund separation theorem:

$$c_1^h = \theta_0^h + \sum_{j \in \mathcal{J}^h}\left(\beta_j + \frac{1}{|H_j|}s_j\right)x_j + \left(y_1^h - \sum_{j \in \mathcal{J}^h}\beta_j^h x_j\right), \ h > F \tag{D.47}$$

$$c_1^h = \theta_0^h + \sum_{j \in \mathcal{J}^h}\left(\beta_j + \frac{1}{|H_j|}s_j\right)x_j + (1-w^h)\left(y_1^h - \sum_{j \in \mathcal{J}^h}\beta_j^h x_j\right), \ h \leq F. \tag{D.48}$$

Taking the variance of these expressions yields

$$\operatorname{var}(c_1^h) = \operatorname{var}(y_1^h) - \sum_{j \in \mathcal{J}^h}(\beta_j^h)^2 + \sum_{j \in \mathcal{J}^h}\left(\beta_j + \frac{1}{|H_j|}s_j\right)^2, \ h > F \tag{D.49}$$

$$\operatorname{var}(c_1^h) = (1-w^h)^2\operatorname{var}(y_1^h) - \sum_{j \in \mathcal{J}^h}(1-w^h)^2(\beta_j^h)^2 + \sum_{j \in \mathcal{J}^h}\left(\beta_j + \frac{1}{|H_j|}s_j\right)^2, \ h \leq F. \tag{D.50}$$

Finally, to obtain the expressions for c_0^h, we take the natural log of equation (D.44) and substitute expression (D.33) or (D.34) for the respective ranges of h. Rearranging the terms, we get

$$c_0^h = -\frac{1}{A}\ln\frac{1}{\pi_0} + E(y_1^h) + \theta_0^h - \frac{A}{2}\operatorname{var}(c_1^h), \ h > F \tag{D.51}$$

$$c_0^h = -\frac{1}{A}\ln\frac{1}{\pi_0} + (1-w^h)E(y_1^h) + \theta_0^h - \frac{A}{2}\operatorname{var}(c_1^h), \ h \leq F. \tag{D.52}$$

Now, all equilibrium quantities are determined in terms of the risk-free asset's price, π_0. To determine this, we take the natural log of equation (D.44) and sum over all agents to obtain

$$H \ln(\pi_0) - A \sum_{h=1}^{H} y_0^h \quad = \quad -A \sum_{h=1}^{H} E(y_1^h) + \frac{A^2}{2} \sum_{h=1}^{H} \mathrm{var}(c_1^h). \tag{D.53}$$

Dividing throughout by H, using the definitions for mean endowments y_0 and y_1 in equation (D.15), and substituting for $\mathrm{var}(c_1^h)$ from equations (D.23) and (D.29), π_0 can be determined in terms of the model's primitive quantities as follows:

$$\pi_0 = \exp\left\{ A\,(y_0 - Ey_1) + \frac{A^2}{2H} \sum_{h=1}^{H} \left[(1 - R_h^2)\mathrm{var}(y_1^h) + \sum_{j \in J^h} \left(\beta_j + \frac{1}{|H_j|} s_j \right)^2 \right] \right\}, \tag{D.54}$$

where

$$R_h^2 \equiv \frac{\sum_{j \in J^h} \left(\beta_j^h\right)^2}{\mathrm{var}(y_1^h)}, \;\; h > F, \quad \text{and} \quad R_h^2 \equiv \frac{\sum_{j \in J^h} (1 - w^h)^2 \left(\beta_j^h\right)^2}{\mathrm{var}(y_1^h)}, \;\; h \leq F \tag{D.55}$$

represent the variability of agent h's endowment that is spanned by the risky assets tradable by the agent.

The competitive equilibrium is now fully determined in closed-form once the supply conditions are substituted:

$$s_0^h = w^h E(y_1^h), \;\; s_j^h = w^h \beta_j^h, \; 1 \leq j \leq J, \;\; s_j = \sum_{1 \leq h \leq F} s_j^h, \; 0 \leq j \leq J \,. \;\; \diamondsuit \tag{D.56}$$