Staircase Formation in Fingering Convection

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Equazioni alle Derivate Parziali nella Dinamica dei Fluidi
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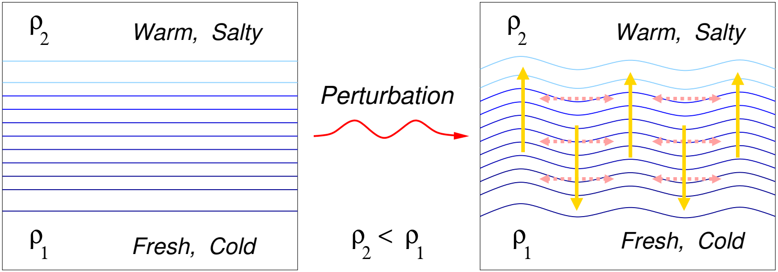
The Fingering Instability (Stern, 1960)

Convection with two scalars:

Salinity (less-diffusing) is *destabilizing*.

Temperature (most-diffusing) is *stabilizing*.



Density is transported *up-gradient*! (light fluid becomes lighter, heavy fluid becomes heavier)

Boussinesq Equations

$$\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} &= -\nabla p + Pr\, Le \left[R_S \underbrace{(R_\rho T - S)}_{B} \hat{\mathbf{z}} + \nabla^2 \mathbf{u} \right] \\
\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T &= Le\, \nabla^2 T \\
\frac{\partial S}{\partial t} + \mathbf{u}\cdot\nabla S &= \nabla^2 S \\
\nabla\cdot\mathbf{u} &= 0
\end{aligned}$$

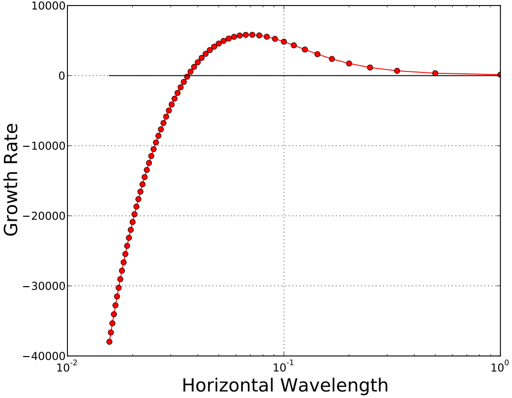
where

$$Pr = \frac{\nu}{\kappa_T}, \quad Le = \frac{\kappa_T}{\kappa_S}, \quad R_T = \frac{g\alpha\Delta T\, H^3}{\nu\kappa_S}, \quad R_S = \frac{g\beta\Delta S\, H^3}{\nu\kappa_S}$$

Density Ratio: $R_\rho = \dfrac{R_T}{R_S}$

Necessary Condition for Fingering Instability: $1 < R_\rho < Le$

Linear Instability: Stern's length



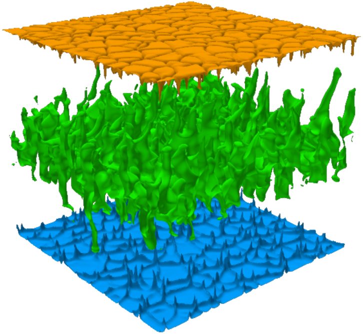
- Small things lose T and S too fast and viscosity wins
- Big things don't lose T efficiently enough
- There's an optimal scale where T is lost, S is retained, and growth of perturbations is maximized.

Stern's length scale:

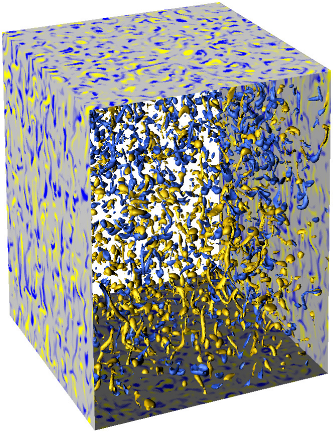
$$l = \left(\frac{\kappa_T \nu H}{g \alpha \Delta T} \right)^{1/4}$$

A Look Inside the Box: From Fingers to Blobs

Salinity, $R_S = 10^9$



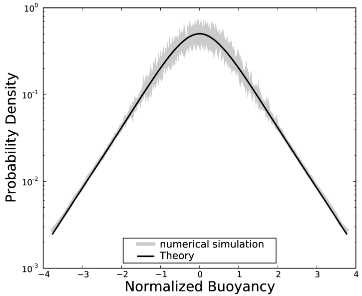
Buoyancy, $R_S = 10^{11}$



Non-Gaussian Statistics

J. von Hardenberg, F.P., Phys. Lett. A, 2010. Owes to V. Yakhot, PRL, **63**, (1989).

3D simulation at $R_S = 10^{11}$, $R_\rho = 1.2$. 2D is the same.



Using a technique due to Yakhot, one may give the following exact expression for the PDF of the buoyancy fluctuations around the horizontal average.

$$P(X) = \frac{E(\chi_B|0)P(0)}{E(\chi_B|X)} \exp\left[-\int_0^X \frac{E(\mathcal{F}_B|y)}{y\,E(\chi_B|y)}\,dy\right]$$

Where:

$$X := \frac{B'}{\langle B'^2\rangle^{\frac{1}{2}}}$$

$$\mathcal{F} := \frac{wB'}{\langle wB'\rangle}; \quad \chi_B := \frac{\nabla B' \cdot ((Le-1)R_\rho \nabla T' + \nabla B')}{\langle \nabla B' \cdot ((Le-1)R_\rho \nabla T' + \nabla B')\rangle}$$

Finger Reynolds Number from Low-R_S Simulations

R_S	$\sigma_{B'}$	σ_W	$\langle W B' \rangle$	l_x	l_v	Re	R_ρ^{loc}
10^8	0.0159	432.2	5.81	0.0402	0.0447	0.58	2.09
10^9	0.0132	1017.6	11.06	0.0218	0.0243	0.74	1.92
10^{10}	0.0099	2621.0	19.45	0.0128	0.0138	1.12	1.70
10^{11}	0.0078	6271.5	33.56	0.0076	0.0070	1.59	1.53
Exp.:	−0.11	0.39	0.25	−0.24	−0.26		

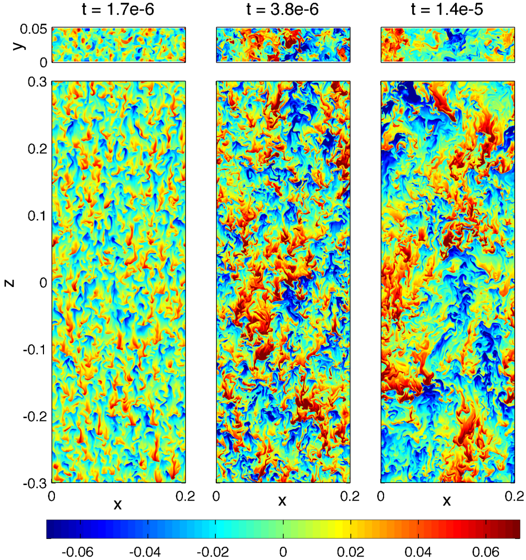
The Reynolds number of a typical blob *increases* with R_S!

Sooner or later it will become non-Stokesian.

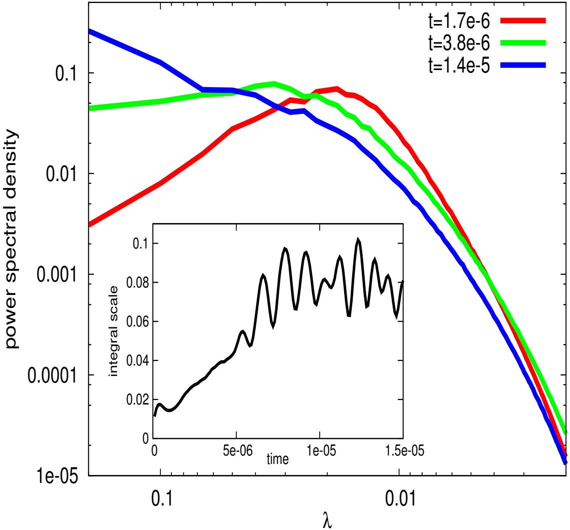
What will happen then?

Salinity Fluctuations at Three Successive Times ($R_S = 10^{13}$; $R_\rho = 1.025$)

F.P., J. von Hardenberg, Phys. Rev. Lett., 2012.



Growth of Horizontal Scales with Time

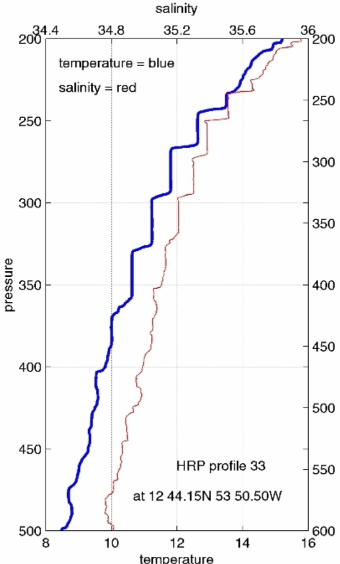


Integral Scale

$$\Lambda = 2\pi \frac{\int k^{-1} E(k)\,\mathrm{d}k}{\int E(K)\,\mathrm{d}k}$$

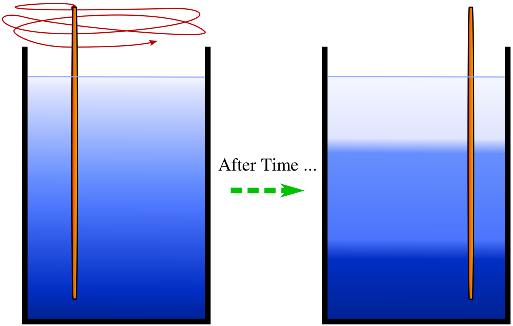
An example of ocean staircases

Data taken during the Salt Fingers Tracer Release Experiment (2001). Courtesy of R. Schmitt, W.H.O.I.



Stirring a Stable Gradient

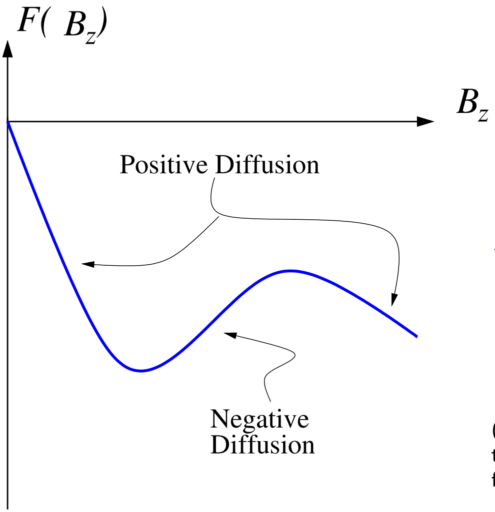
No D-D, just salinity plus a slowly moving rod



Experiments: Park et al. JFM (1994); Ruddick et al. Deep-Sea Res. (1989); Thorpe, JFM (1982).

Theory: Balmforth et al. JFM (1998); Postmentier, JPO (1977); Phillips, Deep-Sea Res. (1972)

Postmentier's Explanation



Conservation Law:

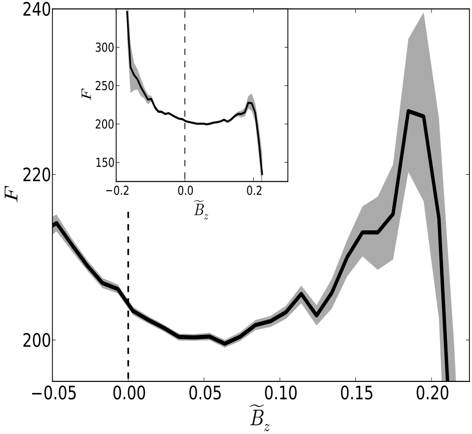
$$\frac{\partial B}{\partial t} = -\frac{\partial F(B_z)}{\partial z}$$

$$\Downarrow$$

$$\frac{\partial B}{\partial t} = \underbrace{-\frac{\partial F(B_z)}{\partial B_z}}_{\text{Diffusion Coefficient}} \frac{\partial^2 B}{\partial z^2}$$

(Yes, this is the same idea as the Perona-Malik's anisotropic diffusion for image denoising!)

Clusters Produce Non-Monotonic Fluxes



- $\widetilde{B}_z < 0$: top-heavy overturning.
- Small $\widetilde{B}_z > 0$: clusters flux downgradient, fingers upgradient.
- Intermediate $\widetilde{B}_z > 0$: fingers dominate.
- Large $\widetilde{B}_z > 0$: thin layer cut-off.

...beautiful but not good enough.
Postmentier's equation has an ultraviolet catastrophe. Ouch!

A Theory for Staircase Formation

F.P., J. von Hardenberg, Acta Appl. Math. (2014). Owes to Balmforth et al. JFM (1998)

Energy equation

$$\bar{e}_t = \left(l\bar{e}^{1/2}\,\bar{e}_z\right)_z + \underbrace{\mathcal{C}}_{\substack{\text{Potential}\\ \text{Energy}\\ \text{Conversion}}} - \underbrace{\mathcal{D}}_{\substack{\text{Kinetic}\\ \text{Energy}\\ \text{Dissipation}}}$$

Buoyancy equation

$$\left.\begin{array}{l}\bar{T}_t = -\left(F_T\right)_z \\ \bar{S}_t = -\left(F_S\right)_z\end{array}\right\} \Rightarrow \bar{b}_t = -\left[\left(\gamma R_\rho - 1\right) F_S\right]_z, \qquad \gamma = \frac{F_T}{F_S}$$

But the fluxes are unknown! ...so I make a minimal recipe:

$$\bar{b}_t = -\left(\underbrace{\mathcal{F}}_{\substack{\text{Constant,}\\ \text{up}-\text{gradient,}\\ \text{fingers flux}}} \quad \underbrace{-l\bar{e}^{1/2}\bar{b}_z}_{\substack{\text{Down-gradient}\\ \text{mechanical}\\ \text{stirring}}} \right)_z$$

$\mathcal{C}$ is the buoyancy flux. $\mathcal{D}$ is a recipe.

Mechanical energy balance of the fluid:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \underbrace{(\bar{e} - z\bar{b})}_{\substack{\text{Kinetic } + \\ \text{Potential} \\ \text{Energy}}} \,\mathrm{d}z = -\int_0^1 \mathcal{D}\,\mathrm{d}z$$

$$\Downarrow$$

$$\mathcal{C} = \text{Buoyancy Flux}$$

On dimensional grounds:

$$\mathcal{D} = A\bar{e}\bar{b}_z^{1/2}$$

(but there could have been a dependence on l, too)

Mixing length: the key ingredient for staircase formation

The (non–constant) diffusivity of both kinetic energy and buoyancy is expressed as the product of a mixing length and a scale of velocity as

$$l\bar{e}^{1/2}$$

Mixing Length:

$$l(\bar{e}) = l_s + \frac{(l_b - l_s)}{1 + \exp(-\eta(\bar{e}^{1/2} - \mu))}$$

is assumed small at low energies (pure fingering) and large at high energies (clusters and well-mixed zones).

The model's equations

See Coclite et al. (submitted, 2018) for existence of global weak solutions

$$\begin{cases} \bar{b}_t & = -\left(\mathcal{F} - l\bar{e}^{1/2}\bar{b}_z\right)_z \\ \bar{e}_t & = \left(l\bar{e}^{1/2}\,\bar{e}_z\right)_z + \mathcal{F} - l\bar{e}^{1/2}\bar{b}_z - A\bar{e}\bar{b}_z^{1/2} \end{cases}$$

Here $\mathcal{F}$ is a *positive* constant!
Too simplistic for a real-life model, but this is just a proof-of-concept.

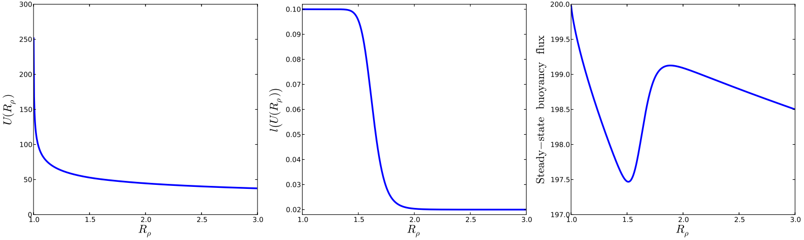
Steady Solutions

...and guess when they're stable and when they're not!

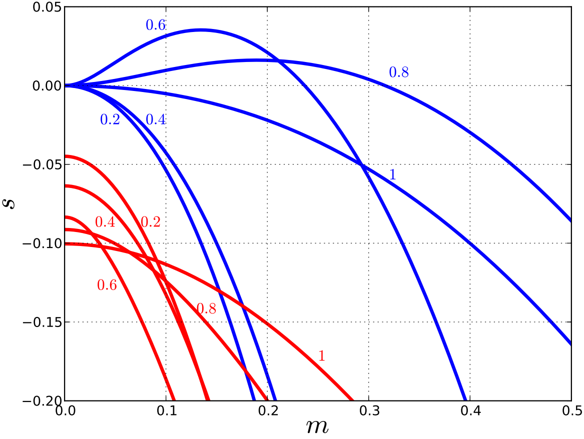
We seek solutions of the form

$$(\bar{b},\, \bar{e}) = \left((R_\rho - 1)z,\, U^2\right)$$

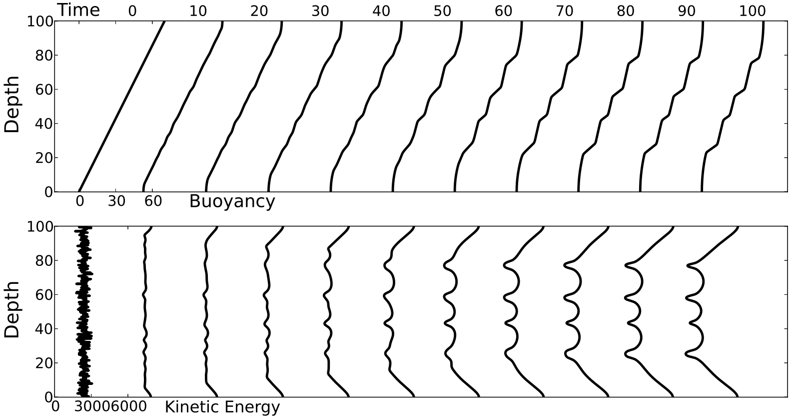
and we obtain



Linear stability analysis of the staircase model



Fully Non-Linear Solutions



Conclusions

- A two–equation embodiment of Postmentier's non–monotonic fluxes idea produces staircase–like profiles.
- Well-posedness of the model is under study (but can be achieved with additional restrictions on diffusive parameterization).
- Much remains to be done, in particular on the long–time evolution of steps.

A Yakhot-Like Theory I

J. von Hardenberg, F.P., Phys. Lett. A, 2010. Owes to V. Yakhot, PRL, **63**, (1989).

Split fluctuations-averages

$$
\begin{aligned}
T(x, y, z, t) &= T'(x, y, z, t) + G_T z \\
S(x, y, z, t) &= S'(x, y, z, t) + G_S z \\
B(x, y, z, t) &= B'(x, y, z, t) + G_B z
\end{aligned}
$$

Equation for buoyancy fluctuations (not closed: contains T')

$$\frac{DB'}{Dt} = (Le - 1)\, R_\rho \nabla^2 T' + \nabla^2 B' - w G_B$$

A Yakhot-Like Theory II

Multiply by B'^{2n-1}, time-volume average $\langle\cdot\rangle$, integrating by parts and obtain

$$(2n-1)\left\langle X^{2n-2}\chi_B\right\rangle = \left\langle X^{2n-2}\mathcal{F}_B\right\rangle$$

where

$$X := \frac{B'}{\langle B'^2\rangle^{\frac{1}{2}}}; \qquad \mathcal{F}_B := \frac{wB'}{\langle wB'\rangle}; \qquad \chi_B := \frac{\nabla B' \cdot ((Le-1)R_\rho \nabla T' + \nabla B')}{\langle \nabla B' \cdot ((Le-1)R_\rho \nabla T' + \nabla B')\rangle}$$

Important: maximum principle for T, $S \Longrightarrow X$ is bounded (no worries about convergence).

The PDF of Buoyancy Fluctuations

Assume space-time averages are the same as ensemble averages.

P p.d.f. of X

$E(\cdot|X)$ expected value of a quantity, given X.

Then

$$(2n-1)\left\langle X^{2n-2}\chi_B\right\rangle = \left\langle X^{2n-2}\mathcal{F}_B\right\rangle$$

$$\Downarrow$$

$$P(X) = \frac{E(\chi_B|0)P(0)}{E(\chi_B|X)} \exp\left[-\int_0^X \frac{E(\mathcal{F}_B|y)}{y\,E(\chi_B|y)}\,dy\right]$$

Symmetries of $E(\chi_B|X)$ and $E(\mathcal{F}_B|X)$

$$P(X) = \frac{E(\chi_B|0)P_X(0)}{E(\chi_B|X)} \exp\left[-\int_0^X \frac{E(\mathcal{F}_B|y)}{y\,E(\chi_B|y)}\,dy\right]$$

$E(\chi_B|X)$ is even, and $E(\chi_B|0) \neq 0$. Expanding around $X = 0$:

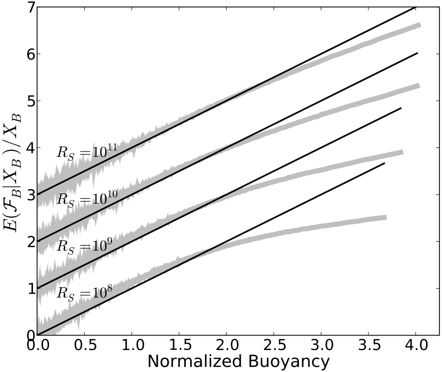
$$E(\chi_B|X) = c + \cdots$$

$E(\mathcal{F}_B|X)$ is even, and $E(\mathcal{F}_B|0) = 0$. Expanding around $X = 0$:

$$\frac{E(\mathcal{F}_B|X)}{X} = X + \cdots$$

N.B. if the $\cdots$ are negligible for large X we get a gaussian distribution!

Conditional Fluxes of Buoyancy Fluctuations



Blobs all of the same size + Stokesian drag gives

$$B' \propto w$$

But the flux is

$$\mathcal{F}_B \propto B'w$$

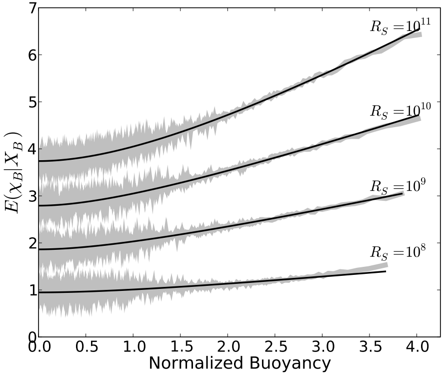
For homogeneous blobs:

$$E(\mathcal{F}_B|X) \propto X^2$$

Flattening at lower R_S because buoyancy patterns still finger-like, not blob-like.

Conditional Dissipation of Buoyancy Fluctuations

see also: F.P. J. von Hardenberg proc. 15th WASCOM conference.



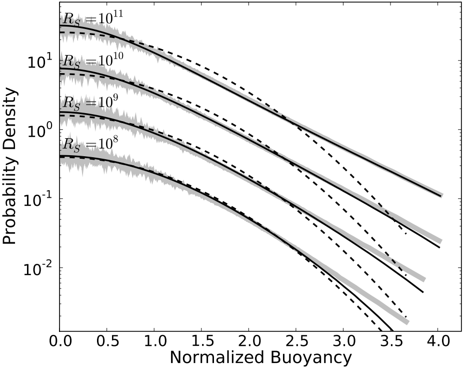
Homogeneous blob cores imply low buoyancy dissipation.

Empirical fit:

$$E(\chi_B|X) = k + \frac{aX^2}{1 + b|X|}$$

Linear tails at high R_S!

Pdf of Buoyancy Fluctuations



Linear tails in $E(\chi_B|X)$ and linear tails in $E(\mathcal{F}_B|X)/X_B$

$\Downarrow$

Exponential tails in the pdf!