Dynamic Bank Capital Regulation in Equilibrium

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Abstract

We study optimal bank regulation in an economy with aggregate uncertainty. Bank liabilities are used as "money" and hence earn lower returns than equity. In laissez faire equilibrium, banks maximize market value, trading off the funding advantage of debt against the risk of costly default. The capital structure is not socially optimal because external costs of distress are not internalized by the banks. The constrained efficient allocation is characterized as the solution to a planner's problem. Efficient regulation is procyclical, but countercyclical relative to laissez faire. We show that simple leverage constraints can get the decentralized economy close to the constrained efficient outcome.

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Introduction

The financial crisis of 2007-2008 emphasized the importance of macroprudential regulation of the banking system. It is widely believed that many banks had inadequate levels of capital prior to the crisis. Regulators and policy makers have been working, at both the national and international levels, to strengthen capital adequacy regulation, but there is still a lack of agreement about the cost of bank capital and how much bank capital is required. The present paper attempts to make a contribution to this debate by studying an equilibrium model of optimal bank capital structure.

Admati and Hellwig (2013) have argued that the seminal paper of Modigliani and Miller (1958) should be the starting point of any discussion of bank capital structure. But, as DeAngelo and Stulz (2015) have pointed out, the framework adopted by Modigliani and Miller is inappropriate as a model of bank capital structure. Whereas corporate debt is a claim on corporate cash flows, bank liabilities circulate as money, in addition to being a claim on cash flows. The liquidity services provided by bank liabilities create a wedge between the risk adjusted returns on bank debt and bank equity. In this sense, debt is a cheaper source of funding than equity.

We use a two-sector dynamic equilibrium model consisting of bankers, producers, and consumers. Bankers raise funds by issuing equity and debt (in the form of deposits) and invest in capital goods. The bankers' capital assets produce revenue using a linear, stochastic technology that is subject to idiosyncratic as well aggregate shocks. Producers have a neoclassical technology with decreasing returns to scale for producing new capital goods. Production is instantaneous and uses only consumption as an input, so there is no need for producers to obtain financing for production. The producers are assumed to maximize profits, which are immediately paid to the owners (consumers).

There is a large number of identical, infinitely lived consumers. Consumers have an initial endowment of capital goods, which they sell to bankers in exchange for debt (deposits) and equity. From that point onwards, consumers receive the returns on debt and equity and manage their portfolios to maximize their expected utility over to an infinite horizon.

One of the key features of our model is the social value of bank deposits. We assume that markets are intertemporally segmented in such a way that deposits are necessary

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for consumption. Income from equity must be converted into deposits before it can be consumed. Because of market segmentation, the conversion of equity income into deposits is time consuming. Delay is costly and creates a wedge between the pecuniary returns earned by deposits and equity. The cost of delay is endogenous and, like other prices, it adjusts to clear markets.

A banker chooses the capital structure that maximizes the market value of the bank at each date. This entails a tradeoff between the funding advantage of debt and the risk of costly default.[1] This tradeoff determines a unique, optimal, capital structure in equilibrium. The optimal capital structure changes over time, of course. Without the funding advantage of debt, 100% equity finance would be optimal; without the risk of costly default, 100% debt finance would be optimal. We need both the funding advantage of debt and costs of default to explain a non-trivial capital structure.

Because markets are incomplete (debt and equity are the only claims on the banks' cash flows), the appropriate efficiency concept is *constrained* efficiency. A constrained efficient allocation can be characterized as the solution to a planner's problem, in which the planner maximizes the expected utility of the representative consumer. In addition to the usual feasibility conditions, the planner is required to use debt and equity to share risk and allocate consumption. This includes the requirement that only deposits can be used to pay for consumption.

In the absence of a negative externality, a laissez faire equilibrium is constrained efficient. We can derive the equilibrium allocation as the solution to the planner's dynamic program and use the first-order conditions for the consumers' problem to "back out" the market clearing prices. The capital structure chosen by the planner turns out to maximize the market value of banks, when debt and equity are priced using the appropriate stochastic discount factors.[2]

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[1]For simplicity, we assume that default results in bankruptcy and the loss of a fixed fraction of the bank's revenue. These costs are internalized by the banker and represent a deadweight loss for the economy.

[2]Even in the absence of a negative externality, the analysis of the laissez fare equilibrium offers a novel perspective from a policy point of view. In an equilibrium model subject to both aggregate and idiosyncratic shocks, it is not obvious what the optimal, dynamic, capital adequacy policy is. How does it respond to shocks? How does it vary over the business cycle? How does it affect capital accumulation and consumption? Solving for a dynamic capital structure that is both privately and socially optimal is a non-trivial problem and offers insights that may be useful for the design of optimal policy in situations where the first best cannot be not achieved.

The rationale for bank regulation is that bank failures impose a negative externality on the economy. These are intended to capture costs of financial distress that are not internalized by the bankers when they choose their capital structure. For simplicity, we assume that the external costs are additively separable from the rest of the economy. This has the technical advantage that the behavior of the bankers, consumers and producers is unaffected by the presence of the externality. Then the equilibrium allocation is still the solution of a dynamic programming problem. The only difference is that the objective function excludes the externalities. In fact, the prices and quantities of the laissez faire equilibrium are the same, with and without the negative externality.

The focus of the paper is optimal, dynamic, capital regulation in an economy subject to aggregate productivity shocks, when bank leverage creates a negative externality. Aggregate uncertainty makes the characterization of the equilibrium and the optimum analytically intractable, so we resort to numerical methods to solve the model globally. It is important to compute a global solution, rather than a linear approximation around a steady state, for example. Non-linearities play an important role in the model and the optimal dynamic policy varies across different states. A good illustration is the behavior of bank leverage over the business cycle. In both the laissez faire equilibrium (*equilibrium*, hereafter) and the optimal solution of the planner's program (*optimum*, hereafter) in the presence of a negative externality, leverage is strongly procyclical. In both cases, an increase in productivity leads to an increase in leverage in the banking sector. This contradicts the view that capital regulation should be countercyclical, that is, requiring more bank capital at the peak of the cycle than at the trough.[3] The relationship between the optimum and equilibrium leverage is highly non-linear, however. When productivity and output are low, the difference between the equilibrium bank leverage and the optimal bank leverage is negligible. When productivity is high, on the other hand, optimal leverage is much lower than equilibrium leverage. In fact, the ratio of optimum leverage to equilibrium leverage decreases monotonically as productivity increases. So, although optimal leverage is procyclical, the policy is countercyclical in the sense that optimal regulation restricts leverage *relatively more* compared to the equilibrium at the top of the cycle than at the bottom. Thus, the model produces two surprising results, the optimality of *procyclical* leverage together with a non-linear *countercyclical* policy of "leaning

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[3]Gersbach and Rochet (2017) provides a rationale for countercyclical policy in this traditional sense.

against the wind." These subtleties would not be observable in a linear approximation of the model, of course.

Another interesting feature of the model's dynamics is what we call the *intertemporal substitution effect*. This is most clearly seen when we shut down the aggregate productivity shocks and observe the transition of the economy from some initial condition to the steady state. Because of the negative externality associated with leverage, we expect leverage to be lower in the optimum than it is in the equilibrium. However, this poses a problem for the planner: reducing leverage will restrict consumption, other things being equal, and increase investment. But what is the point of investing more if the returns cannot be consumed? In fact, it is not optimal to reduce leverage at *every* point on the growth path. Initially, leverage and the bank's default probability are *higher* in the optimum than in the equilibrium. Then, when the capital stock has grown sufficiently large, leverage and the probability of default level off in the optimum, but continue to grow on the equilibrium path and eventually overtake the constrained efficient leverage and default probability. This behavior is a consequence of the fact that the negative externality is linear in the capital stock whereas the marginal utility of consumption is diminishing as the capital stock and consumption grow. When the economy is small, the marginal external cost is relatively unimportant compared to the marginal utility of consumption. As the economy approaches the steady state, the marginal external cost remains the same while marginal utility has become relatively small.

Once we re-introduce aggregate productivity shocks, the model no longer has a steady state, but it has an ergodic set and we can see the intertemporal substitution effect at work here as well. Along the transition to a steady state, it is optimal to increase leverage when capital and consumption are low and increase it when they are high. The same intuition suggests that, in the ergodic set, optimal leverage should be relatively high when productivity is low and relatively low when productivity is high. And this is exactly what we see when we compare leverage in the equilibrium and optimum: the ratio of equilibrium to optimal leverage is countercyclical.

In the presence of the negative externality, the constrained efficient solution to the planner's problem is the best that can be achieved, but it goes beyond what could be interpreted as capital regulation because it requires the regulator to control every aspect of the economy. Unfortunately, there is no way to decentralize decisions about consumption, investment, and portfolios choice, while leaving regulation of bank capital

structure to the central authority. Even if such a decentralization result were available, the fully state-contingent optimal capital regulation might be so complex that it would be unrealistic to expect a regulator, with limited computational ability and information, to implement it.

As an alternative to the central planning solution, we model the behavior of a boundedly rational regulator, while leaving everything else to be determined as in the laissez faire equilibrium, in the form of an ad hoc rule that approximates the optimal leverage policy derived from the planner's problem. In the simplest case, this reduced-form regulation takes the form of a constant, state-independent, upper bound on leverage. A more sophisticated version allows for the upper bound on leverage to vary with aggregate productivity. In this way, we try to mimic a fixed maximum leverage ratio and a pro-cyclical maximum leverage ratio. These constraints on leverage are imposed exogenously on banks, but producers and consumers are not directly constrained. They make their equilibrium decisions in the usual way, taking prices as given, and then prices adjust to clear markets. We refer to an equilibrium relative to an exogenous regulatory rule as a *regulated equilibrium*.[4]

Our analysis of the various regulated equilibria shows that a simple policy can be quite effective in certain circumstances. If we shut down the aggregate productivity shock and set the maximum leverage equal to the steady state leverage in the optimum, the regulated equilibrium with this constant, state-independent maximum leverage looks pretty similar to the optimum. Along the transition path, the leverage constraint is not binding but the unconstrained leverage is close to the constrained efficient leverage. Once we get close to the steady state, the constraint begins to bind and regulated equilibrium is forced to follow the same path as in the optimum.

Although this very simple policy does well if we set the upper bound equal to the constrained efficient leverage in steady state, the results are sensitive to the choice of policy. Choosing a slightly lower upper bound causes the economy to overshoot the constrained efficient steady state and accumulate too much capital. Although this means that consumption will eventually be higher in the steady state, the path as a whole is inefficient and, in the steady state, consumption and welfare could both be raised by reducing the capital stock through depreciation.

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[4]Admittedly, the models of regulation we analyze are suboptimal and ad hoc. Nonetheless, it is interesting to see how close one can come to the first best using such simple, ad hoc rules.

A constant maximum leverage ratio does well in the absence of aggregate uncertainty, but it would be less successful in an economy with aggregate uncertainty. We know that the optimal leverage is strongly procyclical. A constant maximum leverage ratio is either never binding or prevents leverage from rising enough when productivity is high. On the other hand, a state-contingent policy can do quite well. Although the optimum leverage depends on two state variables, the capital stock and the productivity shock, it is sufficient to make the maximum leverage in the regulated equilibrium a function of the productivity shock alone. Setting the maximum leverage equal to the expected leverage in the optimum, conditional on the productivity shock, we find that the regulated equilibrium is very close to the optimum.

One of the advantages of the regulated equilibrium is that we can see the effect of capital regulation on market prices. In particular, we can see how the relative costs of debt equity funding are affected. Again, the cleanest results are obtained for the model in which the aggregate productivity shock is shut down and we consider a constant, state-independent maximum leverage ratio. When the capital stock is low, the returns to debt and equity are quite high and very similar. As the economy grows, approaching the steady state, the returns on debt and equity fall and drift apart. The steady-state returns on equity are determined by the consumers' discount factor, as they would be in any neoclassical growth model. The return on deposits, however, is much lower because of the liquidity premium on deposits. The spread between the returns on debt and equity is also very sensitive to the leverage constraint. A slight tightening of the constraint can drive the return on deposits into negative territory. In the model with aggregate uncertainty, the price dynamics are more complicated, but we continue to see significant differences between the cost debt and equity funding and significant sensitivity to capital regulation.

The rest of the paper is structured as follows. In Section 1 we discuss the contribution of our analysis vis-a-vis the extant literature. In Sections 2-4, we introduce the dynamic equilibrium model of economy with a banking sector and derive the fundamental decentralization results, which are instrumental to our analysis. Section 5 introduces our models of bank regulation. In Section 6 we discuss the numerical results of our models.

1   Literature review

As highlighted in Galati and Moessner (2013), the financial crisis of 2007-09 exposed important shortcomings in our understanding of the nexus between the real economy, the financial system, and monetary policy (Crowe, Johnson, Ostry, and Zettelmeyer 2010, Claessens, Kose, Laeven, and Valencia 2014). Also, externalities play a crucial role in the design of macroprudential policies (De Nicolò, Favara, and Ratnovski 2012), but much of the literature does not consider these externalities or focuses on a representative bank. Our paper fills these gaps proposing the study of optimal, dynamic, capital regulation in an economy subject to aggregate productivity shocks, when bank liabilities circulate as money but leverage of the banking sector creates a negative externality.

There is now a substantial literature on macroeconomic models with financial frictions. Recently, Gertler and Kiyotaki (2015) consider a macroeconomic model of bank runs in which the supply of bank capital is fixed. Thus, the cost of capital does not play an important role in determining the equilibrium capital structure. Similarly, in Brunnermeier and Sannikov (2014), the quantity of bank capital changes due to macro shocks and evolves according to a two-factor stochastic equation. Their model assumes also that the debt risk free. In the models of Gertler and Kiyotaki (2015) and Brunnermeier and Sannikov (2014), the level of consumption is not a choice variable, making prices independent of consumption. These models have the merit of including a stylized financial sector, although without developing a theory of asset pricing. Unlike the macroeconomic literature, our paper focuses on the pricing of debt and equity and the impact on financial decisions, such as the choice of the equilibrium capital structure, rather than the role of financial frictions in the business cycle.

A second strand of literature studies optimal bank capital structure. Allen, Carletti, and Marquez (2015) study a simple general equilibrium model in which banks and firms are funded by debt and equity and choose their capital structures to maximize joint surplus. In equilibrium they show that banks have much higher leverage than firms. Gale and Gottardi (2017) generalize the results of Allen, Carletti, and Marquez (2015), showing that similar results can be obtained in a standard competitive equilibrium model without their restrictive assumptions. They argue that bank leverage can be higher because the equity buffer held by firms does "double duty," making the firms' debt safer and thus making banks safer. In an important quantitative study, Gornal and

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Strebulaev (2015) also study the general equilibrium determination of capital structures in the corporate and banking sectors. They show that for reasonable parameter values, leverage is much higher in the banking sector than in the corporate sector. They argue that banks assets are safer for two reasons, because they are senior claims and because the bank is diversified across firms. In the present paper, we combine the banking and corporate sectors, following the approach in Allen, Carletti, and Marquez (2015). And unlike the static models discussed above, our focus is on the dynamics of capital structure and prices driven by aggregate shocks.

Admati and Hellwig (2013) base their argument that "bank capital cannot be expensive" on the seminal Modigliani and Miller (1958) paper on corporate capital structure. However, banks are special, as DeAngelo and Stulz (2015) point out. Banks provide liquidity services, that is, they are engaged in security design, and cannot take the prices of securities as given. Instead, the banker takes as given the marginal utility that the representative consumer will receive in each state, in each subperiod, and uses the marginal utilities to value the bundle of contingent commodities represented by the securities. In this way, banks provide liquidity insurance to depositors who wish to postpone consumption (Diamond and Dibvig 1983). It is important to note that the optimal level of capital and the leverage structure of the bank along the business cycle is not explored here. In an early paper, Gale (2004) studied the endogenous choice of bank capital structure to provide additional risk sharing to depositors. Again, among the channels of interaction between the banking sector and the real economy, these papers do not consider the role of consumption. An exception is Gale and Yorulmazer (2016), who highlight the social value of deposits and the banks' equilibrium capital structure is determined, similarly to our model, by a trade off between the funding advantages of deposits and the risk of costly default.

A third research strand examines the optimality of bank regulation and, in recent times the welfare effect of capital and liquidity requirements. A first analysis of optimality and the rationality of banking regulation is in Gale and Ozgur (2005).

In order to find an externality that justifies the introduction of capital regulation, one has to go beyond the microeconomic analysis of a single bank and consider the efficiency of risk sharing in the financial sector or the economy as a whole. Financial fragility is one possible justification. A recent review of the literature is provided by Marttynova (2015). She reviews studies exploring how higher bank capital requirements

affect economic growth. The study shows that the way banks meet capital requirements (raise equity, cutting down lending, and reducing asset risk) matters and finds that both theoretical and empirical studies are inconclusive as to whether more stringent capital requirements reduce banks' risk-taking and make lending safer.

De Nicolò, Gamba, and Lucchetta (2014) develop a dynamic model to study the quantitative impact of microprudential bank regulations on bank lending. The model assesses the efficiency and welfare of banks that are financed by debt and equity, undertake maturity transformation, are exposed to credit and liquidity risks, and face financing frictions. They show that the relationship between bank lending, welfare, and capital requirements is concave. More importantly, they argue that resolution policies contingent on observed capital, such as prompt corrective action, dominate in efficiency and welfare terms (non-contingent) capital and liquidity requirements. Relative to this literature, one of our most important contributions is that efficient capital regulation is procyclical and that a state-contingent leverage constraint can get the decentralized economy close to the constrained efficient outcome.

Van den Heuvel (2008, 2016) analyze the welfare cost of capital requirement. His model, like ours, assumes that banks liabilities provide liquidity in an otherwise standard general equilibrium growth setting. He analyzes how capital requirements can affect capital accumulation and the size of the banking sector when there is a tradeoff between the benefit of bank's deposits and the cost of capital requirement and supervision. He argues capital requirements are very costly in terms of welfare (between 0.1 and 1 per cent of the US GDP) because an increase in the capital requirement lowers welfare by reducing the ability of banks to issue deposit-type liabilities. On the other hand, capital requirements reduce bank supervision and the related compliance costs, given the incentive compatibility constraint. While this may be an important factor, in our model we consider a genuine market failure: the typical bank, being small, does not take into account the negative externality of the overall banking sector leverage.

Differently from recent contributions, our model explicitly assumes macroeconomic risk as the main driver. Van den Heuvel (2008, 2016) assumes no aggregate uncertainty. In Boissay, Collard, and Smets (2016), recessions arise from a coordination failure between heterogeneous banks, as opposed to from aggregate uncertainty. They use a simple textbook general equilibrium model, in which banking crises result from the procyclicality of bank balance sheets that originates from interbank market funding. In

their model, a crisis breaks out endogenously, following a credit boom generated by a sequence of small positive supply shocks; it does not result from a large negative exogenous shock. Thus, a procyclical regulation would not be optimal in their setting, as the banks, through the interbank channel, would have problems with reciprocal funding.

Also Phelan (2016) attributes macroeconomic instability to the financial sector. He derives a continuous-time stochastic general equilibrium model in which banks allocate resources to productive projects, and bank deposits provide liquidity services. Bank capital is set with a VaR rule, similarly to Adrian and Shin (2014), that makes leverage procyclical because asset's risk is higher in downturns. Phelan shows that although financial-sector leverage increases social efficiency in the short run, in the long run it increases the frequency and duration of states with bad economic outcomes. Hence, according to Phelan, there is a linkage between procyclicality of leverage and financial instability. While this literature argues that the procyclicality of bank leverage is detrimental to welfare, we show that the leverage in the constrained efficient economy is procyclical.

2 The model

In this section, we introduce the model of competitive equilibrium that provides the framework for the rest of the paper. We characterize the optimal allocation as the solution to a planner's problem and show that the constrained efficient allocation can be decentralized as a laissez faire equilibrium. This approach has two advantages. First, it allows us to compute the equilibrium allocation as the solution to a dynamic programming problem. Second, it implies that the equilibrium allocation is constrained efficient. The equilibrium prices can then be backed out from the first-order conditions of the representative consumer, evaluated at the equilibrium allocation.

Time is assumed to be discrete and is indexed by $t = 0, 1 \ldots$. At each date t, there are two goods, a perishable *consumption* good and a durable *capital* good. The consumption good is used as the sole input for the production of capital goods. The capital good is used as the sole input for the production of consumption goods.

The economy consists of consumers, bankers, and producers. Consumers are the initial owners of capital goods, which they sell to bankers in exchange for deposits and equity. Consumers manage a portfolio of deposits and equity to fund lifetime consumption. They also own the firms that produce capital goods.

Bankers control the technology for producing consumption goods. They fund the purchase of capital goods by issuing debt (deposits) and equity. Constant returns to scale and perfect competition ensure that bankers maximize the market value of their banks but receive no remuneration in return. The bankers pay depositors principal and interest from their revenues. The rest is earnings on equity which can be paid to shareholders as dividends or retained and invested in assets (capital goods). Banks are subject to revenue shocks, which introduce the possibility of default. If a bank has insufficient funds to meet the demand for withdrawals, it is forced into liquidation and settles its debts from the sale of its assets (capital goods).

Producers use a neoclassical technology to produce capital goods. Since production takes place instantaneously and does not involve capital as an input, there is no need for the producers to finance their operations with debt and equity. They choose inputs and outputs to maximize current profits. Profits are immediately distributed to the owners (consumers).

2.1 Market structure

A key assumption in our model is that markets and activities are segmented. The interval $[t, t+1)$, referred to as *period t*, is divided into two *subperiods*, which we call '*morning*' and '*afternoon*.' Some markets are open only in the morning; other markets are open only in the afternoon. This segregation of markets naturally leads to a segregation of activities between the morning and afternoon, as well. The time line is as follows:

- morning of period $[t, t+1)$:

    - the aggregate productivity shock and the bankers' idiosyncratic shocks are realized;
    - bankers' cash flows are realized;
    - consumers withdraw deposits from banks;

- deposits that are not used for consumption can be held until the afternoon;

- afternoon of period $[t, t+1)$:

  - solvent banks pay dividends to shareholders;
  - failed banks are liquidated and their debts settled;
  - new capital goods are produced and sold to banks;
  - banks issue debt and equity to finance the purchase of new capital goods and to optimize their capital structures;
  - consumers purchase new debt and equity and rebalance their portfolios.

This structure forces consumers who want to consume in the morning of period $t+1$ to acquire deposits in the afternoon of period t. A consumer who receives dividends in the afternoon of period t, cannot consume them immediately. Instead, the dividends must be converted into deposits, which cannot be consumed until the morning of period $t+1$.[5]

The segregation of activities between subperiods gives deposits a role as a medium of exchange as well as a store of wealth. Deposits are not simply another asset: they have social value because they make consumption possible. As we shall see, deposits are a cheaper source of funding for banks than equity, because of the liquidity services provided by deposits.

The segregation of activities also explains why banks that are short of 'cash' to pay their depositors cannot obtain additional liquidity by selling part of their capital stock. The market for capital goods is open in the afternoon, but not in the morning.

2.2 Consumers

There is a unit mass of identical and infinitely lived consumers. A consumer begins life with k_0 units of capital goods at date 0 that he sells to bankers in exchange for deposits and equity. We assume there is no consumption or production at date 0, which

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[5]Consumption goods are perishable and cannot be stored between periods. In any case, deposits are more efficient than storage, because bankers invest deposits in productive capital goods, which are productive.

serves only as an opportunity for consumers to sell capital goods and for bankers to buy capital goods and choose their initial capital structure, that is, the amount of deposits and equity they issue in exchange for capital goods.

Consumer preferences are given by the standard, additively separable utility function

$$\sum_{t=1}^{\infty} \beta^t u\left(c_t\right),$$

where $0 < \beta < 1$ is the common discount factor, c_t denotes consumption at date t and $u\left(c_t\right)$ is the utility from consumption c_t. The function $u\left(\cdot\right)$ is assumed to satisfy the usual neoclassical properties:

$$u : \mathbf{R}_+ \to \mathbf{R} \text{ is } C^2 \text{ and } u'\left(c\right) > 0 \text{ and } u''\left(c\right) < 0, \text{ for all } c \geq 0.$$

Consumers manage a portfolio of deposits and equity to provide the optimal consumption stream over their infinite horizon. As we shall see, the return on (fully diversified) deposits is always lower than the return on equity.

2.3   Bankers

There is a unit mass of bankers represented by the interval $[0, 1]$. Each banker $i \in [0, 1]$ receives two productivity shocks at date t, an idiosyncratic shock θ_{it} and a systemic or aggregate shock A_t. One unit of capital produces $\theta_{it} A_t$ units of the consumption good in the morning of period t. We assume that the random variables $\{\theta_{it}\}$ are i.i.d. across i and t. Let $F\left(\theta\right)$ denote the c.d.f. of the random variables $\{\theta_{it}\}$. We assume that F is continuous and increasing on $[0, Z]$, with $F\left(0\right) = 0$ and $F\left(Z\right) = 1$. We assume the shock A_t takes a finite number of values, $A_t \in \mathcal{A} = \{a_1, ..., a_n\}$, and has a stationary transition probability, $p\left(A_{t+1}|A_t\right) > 0$, for every A_t, $A_{t+1} \in \mathcal{A}$. Without loss of generality, we can order the shock values so that $a_1 < a_2 < ... < a_n$.

Because there is a large number of bankers and the productivity shocks are i.i.d., we assume that the cross-sectional distribution of shocks is the same as the probability distribution F. Thus, for any θ, the fraction of banks that receive a shock $\theta_{it} \leq \theta$ is

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$F(\theta)$. In particular, this means that the "law of large numbers" convention is satisfied, so

$$\int_0^Z \theta_{it} di = \mathbf{E}[\theta_{it}],$$

at every date t. Because we are interested in the aggregate behavior of bankers, we drop the subscript i in what follows and use θ_t to denote the generic value of the productivity shock to a representative banker.

Bankers fund the purchase of capital goods by issuing debt (deposits) and equity. One unit of deposits purchased at date t has a face value of $z_t k_t$ at date $t+1$,[6] where each bank is holding k_t units of capital goods at the end of period t.

By issuing debt, the banks expose themselves to the risk of default. In the morning of period $t+1$, a bank produces $\theta_{t+1} A_{t+1} k_t$ units of the good. If the bank has issued deposits with face value less than or equal to $\theta_{t+1} A_{t+1} k_t$, it can redeem the deposits in full. Otherwise, it is in default. In the event of default, the bank incurs additional costs associated with bankruptcy. These costs are assumed to take the form of a fraction $0 < \delta < 1$ of output that is lost when the bank defaults.

2.4 Producers

The technology for producing capital goods is subject to decreasing returns to scale. An input of $I \geq 0$ units of the consumption good produces $\varphi(I)$ units of the capital good instantaneously:

φ is C^1 on $(0, \infty)$, $\varphi'(I) > 0$ and $\varphi''(I) < 0$, for $I > 0$, and $\lim_{I \searrow 0} \varphi'(I) = \infty$.

The production of capital goods is instantaneous, so no finance is required. If producers choose as inputs I_t units of consumption and produce $\varphi(I_t)$ units of capital goods, the revenue is $v_t \varphi(I_t)$, where v_t is the price of capital goods in terms of consumption, and the profit is $v_t \varphi(I_t) - I_t$. The producers maximize profits each period:

$$\pi_t = \sup_{I \geq 0} \{v_t \varphi(I) - I\},$$

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[6]Because of the linearity of the bankers' technology, we scale everything by the size of the capital stock, which allows us to express the equilibrium conditions independently of k_t.

Profits are immediately distributed to the firm's owners (consumers).

2.5   The banker's problem

Competition for capital goods forces bankers to maximize the market value of the securities, debt (deposits) and equity, that they issue. Two bankers with the same capital stock have the same production capabilities and hence the same potential value. The only choice variable they control is their capital structure, which determines the risk of default and the division of returns between debt and equity. The bank with the better capital structure will have a higher market value, which allows the banker to bid more for the available assets (capital goods). In equilibrium, competition drives up the price of capital goods until the market value of the securities issued is just equal to the value of the assets purchased, leaving nothing for the banker himself.

Although a bank may survive for many periods, the banker only needs to look one period ahead when choosing the optimal capital structure. Because the capital structure can be changed at the end of each period, the effect of the banker's choice of capital structure in the afternoon of period t lasts only until the afternoon of period $t + 1$, and the market value of the securities issued depends only on the income earned in period t and the stock of depreciated capital goods that remains at the end of the period.

The capital structure chosen by a banker with k_t units of capital goods is determined by the face value of deposits, $z_t k_t$, issued in the afternoon of period t. The bank will be in default if and only if the revenue, $\theta_{t+1} A_{t+1} k_t$, realized in the morning of period $t + 1$ is less than $z_t k_t$. The bank's total returns consist of the value of deposits in the morning of period $t + 1$, the returns of equity holders in the afternoon of period $t + 1$, and the value of the depreciated capital goods remaining at the end of period $t + 1$. Since the depreciated capital stock, $(1 - \gamma) k_t$, is independent of the banker's decision, it can be ignored for present purposes. Since banks operate subject to constant returns to scale, there is no loss of generality in considering the case of a bank that operates with one unit of capital goods and deposits with face value z_t.

Depositors will diversify their deposits across all banks, thereby eliminating idiosyncratic risk. They are still subject to losses from default, however. A deposit in a bank with an idiosyncratic shock $\theta_{t+1} < z_t/A_{t+1}$ in state A_{t+1} is worth $(1 - \delta) A_{t+1} \theta_{t+1}$. A

deposit in a bank with an idiosyncratic shock $\theta_{t+1} \geq z_t/A_{t+1}$ in state A_{t+1} is worth z_t. The expected value of a deposit in the representative bank, which is equal to the actual yield from a diversified portfolio of deposits, will be

$$A_{t+1} \int_0^{\frac{z_t}{A_{t+1}}} (1-\delta)\,\theta_{t+1}\,dF + z_t \left(1 - F\left(\frac{z_t}{A_{t+1}}\right)\right)$$

in state A_{t+1}. The returns of the equity holders, leaving aside the depreciated capital goods, are

$$A_{t+1} \int_{\frac{z_t}{A_{t+1}}}^Z \left(\theta_{t+1} - \frac{z_t}{A_{t+1}}\right) dF.$$

The characteristics of debt (deposits) and equity issued by the bank are determined by the bank's capital structure. For this reason, the banker cannot take the prices of securities as given. Instead, the banker takes as given the marginal utility of representative consumer in each state and in each subperiod, and uses the marginal utilities to value the securities. For each state $A' \in \mathcal{A}$, let $m_1(A')$ (resp. $m_2(A')$) denote the marginal utility of consumption in the morning (resp. afternoon) of the following period if the productivity shock is A'. The market value of the securities is equal to the weighted sum of the returns on deposits and equity:

$$\sum_{A'} \left[m_1(A') \left(A' \int_0^{\frac{z}{A'}} (1-\delta)\,\theta\,dF + z \left(1 - F\left(\frac{z}{A'}\right)\right)\right) + \right.$$

$$\left. m_2(A')\,A' \int_{\frac{z}{A'}}^Z \left(\theta - \frac{z}{A'}\right) dF \right] p(A'|A).$$

The banker will choose the face value of deposits z per unit of capital to maximize the market value of the bank's securities.

2.6 The consumer's problem

In the afternoon of period t, the representative consumer divides his wealth between deposits and equity. A unit of deposits is a claim on a deposit with face value z_t and a unit of equity is a residual claim on the bank with one unit of capital goods and deposits with face value z_t. The consumer purchases $d_t k_t$ units of deposits at the price q_t and

purchases $e_t k_t$ units of bank equity at the price r_t. One unit of deposits diversified across all banks yields $\lambda_{t+1}(A_{t+1})$ in the morning of period $t+1$, where

$$\lambda_{t+1}(A_{t+1}) = A_{t+1} \int_0^{\frac{z_t}{A_{t+1}}} (1-\delta)\theta_{t+1}dF + z_t \left(1 - F\left(\frac{z_t}{A_{t+1}}\right)\right),$$

in state A_{t+1}. The consumer's budget constraint in the morning of period $t+1$ is

$$c_{t+1} \le d_t \lambda_{t+1}(A_{t+1}) k_t.$$

The consumer chooses $c_{t+1}(A_{t+1})$, subject to the liquidity constraint as part of its maximization problem. The constraint may be binding for some values of A_{t+1} and non-binding for others. When the constraint is not binding, the amount $d_t\lambda_{t+1}(A_{t+1})k_t - c_{t+1}(A_{t+1})$ is carried over to the afternoon and is used to purchase equity or retained as deposit.

Depositors also have a claim on the capital stock of failed banks that is realized in the afternoon of period $t+1$. Let μ_{t+1} denote the value paid by one unit of deposits in the afternoon of period $t+1$, where

$$\mu_{t+1} = \int_0^{\frac{z_t}{A_{t+1}}} \min\left\{z_t - (1-\delta)A_{t+1}\theta_{t+1}, v_{t+1}(1-\gamma)\right\} dF.$$

The depositors receive either the total value of the failed bank's capital, $v_{t+1}(1-\gamma)k_t$, or the difference between the face value of deposits and what they actually received, $z_t - (1-\delta)A_{t+1}\theta_{t+1}$, whichever is less.

In the afternoon of period $t+1$, the equity holders are owners of all of the leftover capital, $(1-\gamma)k_t$, and of the retained earnings of the firms minus what was paid to the depositors in settlement of the bankrupt firms. Let R_{t+1} denote the total return to one unit of equity in the afternoon of period $t+1$. Then the equity holders receive $e_t R_{t+1} k_t$ in the afternoon of period $t+1$, where

$$R_{t+1} = v_{t+1}(1-\gamma) + A_{t+1}\int_{\frac{z_t}{A_{t+1}}} \left(\theta_{t+1} - \frac{z_t}{A_{t+1}}\right)dF - \mu_{t+1}.$$

In the afternoon of period $t+1$, a consumer with deposits $d_t k_t$ receives the settlement $\mu_{t+1}d_t k_t$ from failed banks plus the value of deposits not consumed $\lambda_{t+1}d_t k_t - c_{t+1}$. A

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shareholder with equity $e_t k_t$ has a total return $e_t R_{t+1} k_t$. A consumer also receives the profits from production of capital goods π_{t+1}. Thus, the wealth of a consumer with a portfolio $(d_t k_t, e_t k_t)$ is $\lambda_{t+1} d_t k_t + \mu_{t+1} d_t k_t + R_{t+1} e_t k_t + \pi_{t+1}$. The consumer purchases new deposits $d_{t+1} k_{t+1}$ and equity $e_{t+1} k_{t+1}$ at a cost of $q_{t+1} d_{t+1} k_{t+1} + r_{t+1} e_{t+1} k_{t+1}$, so the budget constraint is

$$q_{t+1} d_{t+1} k_{t+1} + r_{t+1} e_{t+1} k_{t+1} \leq \lambda_{t+1} d_t k_t - c_{t+1} + \mu_{t+1} d_t k_t + R_{t+1} e_t k_t + \pi_{t+1}.$$

The consumer's problem is to choose a sequence $\{(c_t, d_t, e_t)\}_{t=0}^{\infty}$ to maximize

$$\mathbf{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right]$$

subject to the constraints

$$(c_t, d_t, e_t) \geq \mathbf{0} \text{ for any } t,$$
$$c_0 = 0 \text{ and } q_0 d_0 k_0 + r_0 e_0 k_0 \leq k_0,$$
$$c_{t+1} \leq \lambda_{t+1} d_t k_t \text{ for any } t,$$
$$c_{t+1} + q_{t+1} d_{t+1} k_{t+1} + r_{t+1} e_{t+1} k_{t+1} \leq d_t (\lambda_{t+1} + \mu_{t+1}) k_t + e_t R_{t+1} k_t + \pi_{t+1}, \text{ for any } t.$$

2.7 The producer's problem

The producers choose the level of investment $I_t \geq 0$ to maximize profit $v_t \varphi(I_t) - I_t$ at each date and state. The first order condition

$$v_t \varphi'(I_t) - 1 \leq 0 \text{ and } (v_t \varphi'(I_t) - 1) I_t = 0$$

is necessary and sufficient for profit maximization. The Inada conditions ensure that $I_t > 0$ at each date and state, so we can ensure the producers choose the correct value of investment I_t by choosing v_t to satisfy

$$v_t \varphi'(I_t) = 1,$$

at every date and state.

2.8 Equilibrium

An allocation is a non-negative stochastic process $\{(c_t, d_t, e_t, I_t, k_t, z_t)\}$, where at each date t, c_t denotes consumption, $d_t k_t$ is the demand for deposits, $e_t k_t$ is the demand for shares, I_t is investment in new capital goods, k_t is the capital stock and z_t is the face value of deposits supplied by banks. An allocation $\{(c_t, d_t, e_t, I_t, k_t, z_t)\}$ is attainable if

> (i) $c_0 = 0$, and $c_t \leq \lambda_t d_{t-1} k_{t-1}$, for any $t > 0$,
>
> (ii) $(d_t, e_t) = (1, 1)$, for any $t \geq 0$,
>
> (iii) $I_0 = 0$ and $c_t + I_t = A_t k_{t-1} \left(\int \theta dF - \int^{\frac{z_{t-1}}{A_t}} \delta \theta dF \right)$, for any $t > 0$,
>
> (iv) $k_{t+1} = (1 - \gamma) k_t + \varphi(I_t)$, for any $t > 0$.

$$(1)$$

A price system is a non-negative stochastic process $\{(q_t, r_t, v_t)\}$, where at each date t, q_t is the price of deposits, r_t is the price of equity, v_t is the price of capital goods. An equilibrium consists of an attainable allocation $\{(c_t^*, d_t^*, e_t^*, I_t^*, k_t^*, z_t^*)\}$ and a price system $\{(q_t^*, r_t^*, v_t^*)\}$ such that the following conditions are satisfied.

(i) Consumer optimality $\{(c_t^*, d_t^*, e_t^*)\}$ solves the consumer's problem.

(ii) Banker optimality $\{z_t^*\}$ solves the banker's problem at each date t.

(iii) Producer optimality $\{I_t^*\}$ solves the producer's problem at each date t.

3 Constrained efficiency

An attainable allocation is constrained efficient if there is no other attainable allocation that makes the representative consumer better off. In other words, a constrained efficient allocation is an attainable allocation that maximizes the expected utility of the representative consumer subject to the attainability constraints in (1). A constrained efficient allocation can therefore be characterized as the solution of a planner's problem. Because the maximization problem is stationary, it can be put in the form of a recursive and stationary dynamic programming problem. Because period t is divided into two subperiods, morning and afternoon, the planner's problem could begin in either subperiod. For our purposes, it is convenient to think of the planner making his decision

in the afternoon. The state of the system in the afternoon of date t is given by the productivity shock A_t realized in the morning and the capital stock k_t that exists in the afternoon. Given the state (A_t, k_t), the planner chooses the face value of the debt in the banks' capital structure and the next period's consumption, investment, and capital stock. The face value of the debt is determined by z_t, given the state (A_t, k_t). The consumption, investment and capital stock for the next period will all depend on the future productivity shock A_{t+1} and the current state (A_t, k_t). So we denote the consumption, investment and capital stock by $c(A_{t+1})$, $I(A_{t+1})$ and $k(A_{t+1})$, respectively, taking the value of (k_t, A_t) as given. Using this notation, we can state the planner's dynamic programming problem as follows:

$$V(k, A) = \max_{(\mathbf{c}, \mathbf{I}, \mathbf{k}, z)} \sum_{A' \in \mathcal{A}} \beta \{u(c(A')) + V(k(A'), A')\} p(A'|A) \tag{2}$$

subject to the constraints

$$(\mathbf{c}, \mathbf{I}, \mathbf{k}, z) \geq \mathbf{0}, \tag{3}$$

$$c(A') \leq \lambda(A') k, \text{ for any } A' \in \mathcal{A} \tag{4}$$

$$c(A') + I(A') \leq A'k \int \theta dF - A'k \int^{\frac{z}{A'}} \delta \theta dF, \text{ for any } A' \in \mathcal{A} \tag{5}$$

$$k(A') \leq (1 - \gamma) k + \varphi(I(A')), \text{ for any } A' \in \mathcal{A}. \tag{6}$$

Because an increase in the capital stock always increases the value function $V(k, A)$, the constraints (5) and (6) will always holds as equalities. Then the next period's investment is given by

$$I(A') = A'k \int \theta dF - A'k \int^{\frac{z}{A'}} \delta \theta dF - c(A')$$

and the capital stock is given by $k(A') = (1 - \gamma) k + \varphi(I(A'))$, for each A'. This implies that the planner has two non-trivial choices to make. He has to choose the face value of the debt z and divide the total output between consumption and the capital stock.

Proposition 1. *Suppose that* $(\mathbf{c}^*, \mathbf{I}^*, \mathbf{k}^*, z^*) \gg \mathbf{0}$ *is a feasible solution of the planner's dynamic programming problem, that is, it satisfies the constraints (3)–(6) and that the value function* $V(\cdot; A')$ *is concave and C^1 for each A'. Then* $(\mathbf{c}^*, \mathbf{I}^*, \mathbf{k}^*, z^*)$ *is an optimal*

solution of the planner's problem defined by (2)–(6) if and only if it satisfies the first order conditions

$$\beta u' \left(c^* \left(A' \right) \right) = \ell_1^* \left(A' \right) + \ell_2^* \left(A' \right) \varphi' \left(I^* \left(A' \right) \right), \ \forall A' \in \mathcal{A}, \tag{7}$$

$$\beta \frac{\partial}{\partial k} V \left(A', k^* \left(A' \right) \right) = \ell_2^* \left(A' \right), \ \forall A' \in \mathcal{A}, \tag{8}$$

$$\sum_{A'} \beta u' \left(c^* \left(A' \right) \right) \frac{\delta z^*}{A'} F' \left(\frac{z^*}{A'} \right) p \left(A'|A \right) = \sum_{A'} \ell_1^* \left(A' \right) \left(1 - F \left(\frac{z^*}{A'} \right) \right) p \left(A'|A \right), \tag{9}$$

for some positive multipliers $\ell_1^* \left(A' \right)$ and $\ell_2^* \left(A' \right)$.

Proof. All proofs are collected in the appendix. $\square$

4 Recursive equilibrium

In this section we show that the solution to the planner's dynamic programming problem can be decentralized as a competitive equilibrium. Because the planner's problem is recursive, the equilibrium will also be recursive. The planner's problem determines the values of consumption, investment, the capital stock, and the capital structure parameter at each date, in each state. To specify the attainable allocation for a recursive equilibrium, we just have to set $d_t = e_t = 1$ for each date and state. Then it remains to specify the prices (q_t, r_t, v_t) for each date and state so that consumers, bankers and producers solve their respective optimization problems by choosing the appropriate quantities.

4.1 The consumer's problem

The first step is to show that the prices of deposits and equity can be chosen so that consumers choose $d_t = e_t = 1$ at each date and state. The state at date t is (k, A), where k is the capital stock carried forward to date $t + 1$ and A is the productivity of capital in date t. The state at date $t + 1$ is denoted (k', A'). As usual, q and r are the respective prices of deposits and equity and d and e are the respective quantities of debt and equity chosen, in the afternoon of date t. We suppress the reference to the initial state (k, A) in what follows, but obviously the prices, q and r, and the quantitities, e and

d, are functions of the initial state. We introduce a variable s to represent the amount of wealth the consumer has to invest at the end of date t. In equilibrium, $s = vk$, but the consumer does not take this into account.

The consumer's objective function in the afternoon of date t is

$$\sum_{A_{t+1}} \left\{ \beta u \left(c \left(A' \right) \right) + \beta V \left(s \left(A' \right) ; k \left(A' \right) , A' \right) \right\} p \left(A' | A \right) ,$$

where $V \left(s \left(A' \right) ; k \left(A' \right) , A' \right)$ is the expected utility of a consumer with wealth $s \left(A' \right)$ in the afternoon of date $t + 1$, when the state is $\left(k \left(A' \right) , A' \right)$. In the afternoon of date t, a consumer with wealth s chooses a portfolio $\left(dk, ek \right)$, consisting of dk units of deposits and ek units of equity. One unit of deposits purchased in the afternoon of date t pays $\lambda \left(A' \right) k$ in the morning of date $t + 1$ and $\mu \left(A' \right) k$ in the afternoon of date $t + 1$. One unit of equity yields $R \left(A' \right) k$ units of goods in the afternoon of date $t + 1$. The portfolio $\left(dk, ek \right)$ yields consumption

$$c \left(A' \right) \leq \lambda \left(A' \right) dk$$

in the morning of date $t + 1$ and wealth

$$\mu \left(A' \right) dk + R \left(A' \right) ek$$

in the afternoon of date $t + 1$.

The consumer's decision problem in the afternoon of t is to choose $\left(d, e, \mathbf{c}, \mathbf{s} \right)$ to solve

$$V \left(s; k, A \right) = \min_{\left(d, e, \mathbf{c} \left(A' \right) , \mathbf{s} \left(A' \right) \right)} - \sum_{A'} \left\{ \beta u \left(c \left(A' \right) \right) + \beta V \left(s \left(A' \right) ; k \left(A' \right) , A' \right) \right\} p \left(A' | A \right) \quad (10)$$

subject to the constraints

$$qdk + rek - s \leq 0 \tag{11}$$
$$c \left(A' \right) - \lambda \left(A' \right) dk \leq 0 \tag{12}$$
$$c \left(A' \right) + s \left(A' \right) - \left(\lambda \left(A' \right) + \mu \left(A' \right) \right) dk - R \left(A' \right) ek - \pi \left(A' \right) \leq 0. \tag{13}$$

We assume that $V \left(\cdot; k, A \right)$ is concave and C^1, so that the optimal portfolio is determined by the first order conditions of the consumer's problem.

Proposition 2. *At any state (k, A), consumers will choose $d = e = 1$ if and only if*

$$m_0 q = \sum_{A'} \left\{ m_1 \left(A'\right) \lambda \left(A'\right) + m_2 \left(A'\right) \mu \left(A'\right) \right\} p \left(A'|A\right)$$

and

$$m_0 r = \sum_{A'} m_2 \left(A'\right) R \left(A'\right) p \left(A'|A\right)$$

where m_0 is the marginal utility of income in the afternoon when the decision is made and $m_1 \left(A'\right)$ (resp. $m_2 \left(A'\right)$) is the marginal utility of income in the morning (resp. afternoon) of the subsequent period, when state A' occurs.

These equations determine the equilibrium prices and ensure that the consumers' demand for deposits and equity clear the market, at each date and state.

4.2 The bankers' problem

From Proposition 1, the planner's choice of z is determined by the first order condition

$$\sum_{A'} \beta u' \left(c \left(A'\right)\right) \frac{\delta z}{A'} F' \left(\frac{z}{A'}\right) p \left(A'|A\right) = \sum_{A'} \ell_1 \left(A'\right) \left(1 - F \left(\frac{z}{A'}\right)\right) p \left(A'|A\right),$$

where $\ell_1 \left(A'\right)$ is the Lagrange multiplier on the constraint (4). The first order condition for the banker's problem is characterized in the following proposition.

Proposition 3. *For any values of $m_1 \left(A'\right) > 0$ and $m_2 \left(A'\right) > 0$, $A' \in \mathcal{A}$, the solution of the banker's problem satisfies the first order condition*

$$\sum_{A'} \left[m_1 \left(A'\right) \frac{\delta z}{A'} F' \left(\frac{z}{A'}\right) - \left(m_1 \left(A'\right) - m_2 \left(A'\right)\right) \left(1 - F \left(\frac{z}{A'}\right)\right) \right] p \left(A'|A\right) = 0$$

The solution is uniquely determined if the summand

$$m_1 \left(A'\right) \frac{\delta z}{A'} F' \left(\frac{z}{A'}\right) - \left(m_1 \left(A'\right) - m_2 \left(A'\right)\right) \left(1 - F \left(\frac{z}{A'}\right)\right)$$

is increasing in z, for each $A' \in \mathcal{A}$.

Comparing the first order condition from Proposition 3, with the first order condition from the planner's problem, we see that the two conditions are identical since

$$\beta u' \left(c \left(A' \right) \right) = m_1 \left(A' \right)$$

and

$$\ell_1 \left(A' \right) = m_1 \left(A' \right) - m_2 \left(A' \right).$$

If the monotonicity condition of Proposition 3 is satisfied, this ensures that the bank will choose the correct value of z at each date and state.

4.3   The producer's problem

To induce the producers to produce the right amount of capital goods, it is sufficient to set the price of capital goods so that $v \left(A' \right) \varphi' \left(I \left(A' \right) \right) = 1$ for every A'. But in order to show that the equilibrium can be decentralized, we need to check that this definition is consistent with our definition of q and r.

Proposition 4. *If q and r are defined by the first order conditions in Proposition 2 and $v = 1/\varphi' \left(I \right)$, where I is the investment in state (k, A), then $v = q + r$.*

This shows that our definitions of prices are consistent and the optimal solution of the planner's problem can be decentralized as a competitive equilibrium.

5   Regulated equilibrium

While the constrained efficient solution to the planner's problem shows the best that can be achieved, it is not a realistic model of regulation. First, it requires the regulator to control every aspect of the economy. There is no decentralization theorem to show how decisions about consumption, investment, and capital structure could be left in the hands of private decision makers, subject to regulation by a central authority. Second, the solution to the planner's problem may be so complex that it would be unrealistic to expect it to be implemented by a regulator with limited information and computational ability.

These two observations lead to us consider a reduced form model of a boundedly rational regulator. Instead of trying to implement the constrained efficient allocation, we assume the regulator imposes an upper bound on bank leverage, while leaving everything else to be determined as in the laissez faire equilibrium. The limitations on the regulator's information and computational ability are captured by an ad hoc rule that approximates, more or less closely, the leverage policy that is part of the solution to the planner's problem. In the simplest case, we assume the regulator chooses a fixed upper bound on leverage, denoted by $\bar{z}$. A more sophisticated regulator might be able to choose the upper bound, $\bar{z}(A)$, as a time invariant function of the productivity shock A. Finally, the most sophisticated model of the regulator allows the upper bound $\bar{z}(k, A)$ to be a time invariant function of the state (k, A). Producers and consumers are not directly affected by this regulation; only the banker's behavior is directly constrained. Apart from the leverage constraint, the banker's decision problem is essentially the same as described in Section 2.5. In each state (k, A), the banker chooses the value of leverage z to maximize

$$
\sum_{A'} \left[m_1(A') \left(A' \int_0^{\frac{z}{A'}} (1 - \delta) \theta dF + z \left(1 - F \left(\frac{z}{A'} \right) \right) \right) + \right.
$$
$$
\left. m_2(A') A' \int_{\frac{z}{A'}}^{Z} \left(\theta - \frac{z}{A'} \right) dF \right] p(A'|A)
$$

subject to the constraint $z \leq \bar{z}(k, A)$, where $\bar{z}(k, A)$ is exogenously given.[7] Apart from this change in the banker's problem, the definition of a regulated equilibrium is essentially the one given in Section 2.8.

To calculate the regulated equilibrium, we use a variation of the method described in Section 4. We begin by setting up a planner's problem to represent the behavior of consumers, producers and bankers. As in Section 4, the "planner" ignores the negative externality. The difference, now, is the "planner" is subject to the upper bound on leverage, $\bar{z}(k, A)$, which he treats as an exogenous constraint. The addition of the

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[7]To simplify the discussion, we deal explicitly with the case where the upper bound is contingent on the state (k, A). The other two cases, where the upper bound is either a constant $\bar{z}$ or a function $\bar{z}(A)$ of the productivity shock, are obtained as special cases.

leverage constraint is, in fact, the only change in the planner's problem. Thus, the planner's dynamic programming problem is as follows:

$$V\left(k, A\right) = \max_{(\mathbf{c},\mathbf{I},\mathbf{k},z)} \sum_{A' \in \mathcal{A}} \beta\left\{u\left(c\left(A'\right)\right) + V\left(k\left(A'\right), A'\right)\right\} p\left(A'|A\right)$$

subject to the constraints

$$\left(\mathbf{c}, \mathbf{I}, \mathbf{k}, z\right) \geq \mathbf{0},$$

$$z \leq \bar{z}\left(k, A\right),$$

$$c\left(A'\right) \leq \lambda\left(A'\right) k, \text{ for any } A' \in \mathcal{A},$$

$$c\left(A'\right) + I\left(A'\right) \leq A'k \int \theta dF - A'k \int^{\frac{z}{A'}} \delta\theta dF, \text{ for any } A' \in \mathcal{A},$$

$$k\left(A'\right) \leq \left(1 - \gamma\right) k + \varphi\left(I\left(A'\right)\right), \text{ for any } A' \in \mathcal{A}.$$

Notice that the value function $V\left(k, A\right)$ depends only on the state (k, A) because the leverage constraint $\bar{z}\left(k, A\right)$ is independent of time. The solution of this dynamic programming problem provides us with a stochastic process $\{(c_t, I_t, k_t, z_t)\}$ which can then be decentralized using the same method as in Section 4, that is, by backing out the equilibrium prices $\{(q_t, r_t, v_t)\}$ using the first-order conditions for the consumer's and producer's problems.

For our numerical results, we have investigated four versions of the regulated equilibrium. In the first two, the maximum leverage is assumed to be a constant, $\bar{z}$, independent of k and A. In these simulations, we assume that $\bar{z} = 0.55$ and $\bar{z} = 0.53$. The first value is the average steady value of leverage in the constrained efficient allocation, when there is no aggregate uncertainty. The slightly lower value, $\bar{z} = 0.53$, is used to test the sensivity of the equilibrium to a tightening of the constraint. The non-contingent policy does a reasonable job in the model with no aggregate uncertainty, but obviously will not do as well when there are aggregate productivity shocks. In the third experiment, we allow the maximum leverage to be a function $\bar{z}\left(A\right)$ of the aggregate productivity shock A. The value of $\bar{z}\left(A\right)$ is set equal to the conditional mean leverage in the constrained efficient allocation, for each value of A. In the fourth model of regulation, we allow the maximum leverage to be a function $\bar{z}\left(k, A\right)$ of the state (k, A).

Although we refer to the rules $\bar{z}$, $\bar{z}(A)$, and $\bar{z}(k, A)$ as "models" of the regulator, there is in fact no optimization going on. We have simply chosen rules that approximate the leverage observed in the constrained efficient allocation. Even if we take as given the form of the rule, for example, assume that maximum leverage is a constant $\bar{z}$, the policy may not be optimal in a decentralized economy. Consumers, bankers, and producers are making decisions that may be sub-optimal. A regulator who takes into account the impact of $\bar{z}$ on private decisions (about consumption, investment, etc.) might be able to do better. For all these reasons, the "models" of regulation we are analyzing here are quite ad hoc. Nonetheless, it is interesting to see how close one can come to the first best using such simple, ad hoc rules.

6 Results

In our economy, the motivation for bank regulation is provided by a negative externality created by the overall leverage of the banking sector. The externality is incorporated in the utility function of the representative agent,

$$u(c) - \xi z k,$$

where $\xi > 0$ is a parameter and zk measure the overall leverage of the banking sector. Because the cost of the externality is additively separable, it affects the consumer's welfare but does not affect the behavior of the consumers, bankers, and producers. Thus, it does not change the definition of equilibrium. This fact is important because it allows us to calculate equilibrium behavior and prices in the same way, whether there is an externality or not. The cost is fully internalized, on the other hand, in the social planner's optimal allocation. As we have seen, the regulator can only restrict the leverage of the banks, so his ability to achieve a constrained efficient allocation is limited and will generally fall short of restoring the constrained efficient allocation.

The results we present in this section are based on a numerical solution of the equilibrium in different regulatory scenarios. To solve the problem, we specify the model as

follows. The utility function, including the negative externality induced by the leverage of the banking sector, is

$$u(c) = \frac{c^{1-\alpha} - 1}{1 - \alpha} - \xi z k \quad \text{for } \alpha \in]0, 1[\quad \text{and} \quad u(c) = \log c - \xi z k \quad \text{for } \alpha = 1,$$

and the production function is $\varphi(I) = I^\eta$ for $\eta \in]0, 1[$. We assume that the distribution of θ is a generalized uniform with cumulative function $F(\theta) = \left(\frac{\theta}{Z}\right)^m$, with $m \geq 1$ and support $[0, Z]$. The macroeconomic shock A_t is assumed to follow a log-AR(1) process,

$$\log A_{t+1} = \rho \log A_t + \sigma \varepsilon_{t+1},$$

where ε_t are i.i.d. shocks drawn from a truncated Normal distribution $\mathcal{N}(0, 1)$. Denote $[\underline{A}, \overline{A}]$ the support of A, where the bounds are set at three times the unconditional standard deviation of $\log A$.

We solve the regulator's dynamic program using a value function iteration approach on a discrete grid of states. In particular, for the case with no aggregate uncertainty, we set $n_k = 601$ levels of k_t as

$$\left\{ \overline{k}, \overline{k}(1-\gamma)^{1/16}, \overline{k}(1-\gamma)^{2/16}, \ldots, \overline{k}(1-\gamma)^{n_k/16} \right\}$$

with $\overline{k} = 22$, while the optimization is based on $n_z = 601$ discrete levels of z in the interval $[0, Z]$.[8] For the model with aggregate uncertainty, we discretize A_t with $n_A = 5$ points using the method of Tauchen (1986). The optimization with respect to contingent consumption is made by an exhaustive search over a discrete grid of consumption levels with $n_C = 201$ points. Due to the increased size of the grid of states in the case with aggregate uncertainty, we adopt a discretization based on $n_k = 151$ levels of k_t

$$\left\{ \overline{k}, \overline{k}(1-\gamma)^{1/2}, \overline{k}(1-\gamma)^{2/2}, \ldots, \overline{k}(1-\gamma)^{n_k/2} \right\}$$

with $\overline{k} = 29$ and in the optimization $n_z = 121$ discrete levels of z_t.

The numerical results are based on the baseline parameters $Z = 1$, $m = 5$, $\rho = 0.90$, $\sigma = 0.04$, $\beta = 0.95$, $\delta = 0.10$, $\eta = 0.48$, $\gamma = 0.12$, $\alpha = 0.70$, and $\xi = 0.07$. While we

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[8]In the case with no aggregate uncertainty, the liquidity constraint in (4) holds as equality, and so optimal consumption is determined by the choice of z.

choose these parameters for the purpose of illustrating the properties of the model, the qualitative conclusions we draw here below are general.

6.1 Intertemporal substitution

We begin by looking at some features of the model that are most easily seen and understood in the context of an economy with no aggregate uncertainty. For this purpose we set the productivity parameter $A = 1$ at all dates. One interesting feature of the model that can be illustrated in this case is the intertemporal substitution effect. Because of the negative externality associated with high leverage, we intuitively expect that leverage will be too high under laissez faire, but this is not always true. There are times when the optimal leverage will be lower than the laissez faire level and times when it will be higher.

Figure 1 shows the policy functions for leverage, consumption and investment. These policy functions are the solutions of the dynamic programming problems corresponding to the laissez faire equilibrium and the constrained optimum. We can see that there is a capital level $\hat{k}$ such that,[9] for values of k below $\hat{k}$, leverage is higher in the constrained optimum and lower in the laissez faire equilibrium. For values of k higher than $\hat{k}$, the inequality is reversed. Correspondingly, consumption is higher and investment is lower in the constrained optimum for k less than $\hat{k}$. Conversely, consumption is lower and investment higher in the constrained optimum for k greater than $\hat{k}$. The intuition for this is simple. The negative externality is a linear function of deposits, whereas utility is concave in the level of consumption. When the capital stock is low (resp. high), the marginal utility of consumption is high (resp. low), whereas the marginal negative externality of deposits is constant. For this reason, the optimal policy increases leverage relative to the laissez faire equilibrium when capital is low and reduces leverage when capital is high.

The comparison of policy functions does not tell the whole story, however, because the economy grows at different rates under laissez faire and the constrained optimum, so the capital stocks will be different. Figure 2 shows the long run growth paths of the laissez faire and constrained efficient economies. Initially, the capital stock grows faster

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[9]While in the current choice of parameters $\hat{k} = 16$, the behaviour we describe is general.

under laissez faire but, eventually, it is overtaken by the constrained optimum. We can see the reason for this if we look at leverage. There is a date $\hat{t}$ (in our case equal to 20), such that leverage is higher under the constrained efficient policy for $t < \hat{t}$ and it is lower for $t > \hat{t}$. As a result, consumption is initially higher in the constrained optimum, but the laissez faire economy overtakes it because its capital stock is growing faster. When leverage on the constrained efficient path drops below leverage on the laissez faire path, however, the constrained optimum starts to grow faster and eventually its consumption overtakes consumption on the laissez faire path. This guarantees that, in the long run, the capital stock and consumption will both be higher under the constrained efficient policy, even though leverage is lower than in the laissez faire equilibrium. In short, the constrained efficient policy trades lower consumption in the medium run for higher consumption in the short and long run.

Figures 3 and 4 show the corresponding policy functions and growth paths for the laissez faire equilibrium and a variety of regulated equilibria. In a regulated equilibrium, the maximum leverage $\bar{z}$ is imposed exogenously, but the remaining endogenous variables are determined, in the usual way, by optimizing behavior and market clearing conditions. In one case, we set the maximum leverage $\bar{z}$ equal to 0.55, the leverage attained in the constrained efficient steady state. In another, we reduce the maximum leverage to 0.53, to test the sensitivity of the equilibrium to regulation. In a third, we assume the maximum leverage is a function $\bar{z}(k)$ of the capital stock and set it equal to the leverage chosen in the constrained efficient allocation.[10] In the first two cases, the leverage constraint does not bind until the equilibrium is close to the steady state. In the third case, the equilibrium path approximates the constrained efficient path. One interesting feature is that the tighter regulation $\bar{z} = 0.53$ leads to larger deviations from the optimum than the looser constraint $\bar{z} = 0.55$. In particular, the steady state capital stock shown in Figure 4 is much larger than the other three, which are relatively close together. This suggests that relatively small changes in the maximum leverage can have large effects in the long run. The excess capital accumulation, which occurs because of the restriction of consumption, is inefficient in two ways. First, consumption is lower along the transition to the steady state; second, although consumption is slightly higher in the steady state, most of the extra output goes to investment required to replace depreciating

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[10]Although the maximum leverage $z(k)$ equals the actual leverage in the constrained efficient case, the other endogenous variables may be distorted. In other words, the regulated equilibrium will not necessarily be constrained efficient.

capital: although the steady state capital stock is 5% higher on the tightly regulated path, compared to the constrained efficient path, consumption is only 2% higher, while investment is 10% higher. Welfare could be increased by reducing investment, allowing the capital stock to depreciate, and increasing consumption. By contrast, the steady state consumption levels on the other growth paths (laissez faire, loosely regulated, and contingently regulated) are almost identical to the constrained efficient path.

6.2 Cost of debt and equity

The model allows us to characterize the relative cost of debt and equity. Figure 5 shows the net returns on debt and equity along the laissez faire equilibrium path. Both are quite high when the capital stock is low and fall progressively as capital stock rises. This fall in rates of return cannot be due to diminishing returns in the banking sector, because the bank technology is linear. Instead, it results from the slowing rate of growth dictated by diminishing returns to scale in the production of capital goods. The (gross) return on equity is denoted by $1 + \rho_t$ and defined by

$$1 + \rho_t = \frac{R_{t+1}}{r_t}$$

where r_t is the price of equity in the afternoon of period t and R_{t+1} is the total return to equity in the afternoon of period $t + 1$. As we showed in Proposition 2, the price of equity satisfies the following first order condition for the consumer's problem

$$\psi_t r_t = \beta \psi_{t+1} R_{t+1},$$

where ψ_t denotes the marginal utility of one unit of the consumption good in the afternoon of date t.[11] Solving for r_t and substituting the expression in the equation for ρ_t, we obtain

$$1 + \rho_t = \frac{\psi_t}{\beta \psi_{t+1}}. \tag{14}$$

The rates of return on equity are illustrated in Figure 6. The returns on equity are similar for the laissez faire equilibrium and the three regulated equilibria. All four

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[11]In the statement and proof of Proposition 2, we used the Lagrange multipliers m_0 and m_2 to indicate ψ_t and ψ_{t+1}, respectively.

returns are highest when the capital stock is lowest and decline over time as the capital stock increases and the economy converges to a steady state. The decline in the rates of return are a corollary of the slowing growth rate as the as the economy approaches the steady state. When consumption is low and the growth rate is high, we expect ψ_t/ψ_{t+1} to be high and when consumption is high and the growth rate is low, we expect ψ_t/ψ_{t+1} to be low. Then equation (14) implies a fall in the return on equity. As the economy converges to the steady state, ψ_t/ψ_{t+1} will converge to one and equation (14) implies that $1 + \rho$ will converge to $1/\beta$. This is what happens in any neoclassical growth model, of course, so there is nothing surprising in this. In Figure 6, we see that the return on equity converges to 0.05, which is consistent with our parameterization $\beta = 0.95$.

The yields on deposits are a different story, however. Although the deposit rate is lower than the return on equity and declines over time in each case, there are noticeable differences even in the limit. The behavior of the return on deposits can be explained by two factors, the rate of growth and the tightness of the leverage constraint. The (gross) return on debt is denoted by $1 + i_t$ and defined by

$$1 + i_t = \frac{\lambda_{t+1} + \mu_{t+1}}{q_t}$$

where q_t satisfies the first order condition from the consumer's problem shown in Proposition 2,

$$\psi_t q_t = \beta \left\{ u'(c_{t+1}) \lambda_{t+1} + \psi_{t+1} \mu_t \right\}.$$

In our simulations, the value of μ_{t+1} is very small because the probability of a bank defaulting is quite low, generally around 5%. If we ignore this term, the yield on deposits can be approximated by

$$1 + i_t \approx \frac{\lambda_{t+1} \psi_t}{\beta u'(c_{t+1}) \lambda_{t+1}} = \frac{\psi_t}{\beta u'(c_{t+1})}.$$

The expression on the right is illustrated in Figure 7. Comparing the graph of i_t in Figure 6 with the graph of $\psi_t / \beta u'(c_{t+1})$ in Figure 7, this seems to be a reasonable approximation.

When the liquidity constraint is not binding, $\psi_t = u'(c_t)$, and

$$1 + i_t \approx \frac{u'(c_t)}{\beta u'(c_{t+1})}.$$

In this case, which occurs when the capital stock is low and the growth rate is relatively high, the return on deposits and equity will be similar. As the economy grows and gets closer to the steady state, the leverage constraint begins to bind, $\psi_t < u'(c_t) \approx u'(c_{t+1})$, and so the return on deposits, $1+i$, will fall below the return on equity, $1+\rho$. The tighter the leverage constraint, the lower the return on deposits. The most striking feature is that the deposit rate in the regulated equilibrium with $\bar{z} = 0.53$ is negative and much lower than the rates in the other three cases. Again, the interest rate i_t seems to be very sensitive to a (modest) tightening of the upper bound on leverage, whereas consumption does not differ much between the equilibria with $\bar{z} = 0.55$ and $\bar{z} = 0.53$ because the tighter leverage constraint leads to an inefficiently higher investment and capital stock.

6.3   Capital structure dynamics

In the version of the model with no aggregate productivity shocks—and hence no aggregate uncertainty—fixing the maximum leverage at $\bar{z} = 0.55$ is a reasonably good approximation to an optimal policy (Figure 3). But once aggregate productivity shocks are introduced, we must allow for contingent regulation. Figure 8 illustrates the constrained efficient and laissez faire paths for a sequence of aggregate shocks. Two features of the simulation are very clear. First, leverage is procyclical in the sense that an increase in productivity (A) leads to an increase in leverage (z). This is true for both the laissez faire equilibrium and the constrained efficient allocation.[12] Second, the constrained efficient leverage is always lower than the laissez faire equilibrium. This is quite intuitive because of the presence of the negative externality, which the planner internalizes and the bankers do not.

In fact, we can go further and say that the constrained efficient policy is "countercyclical," in the following sense: the constrained efficient leverage is proportionately smaller compared to the laissez faire leverage in the upswing of the cycle, when A is high, than it is in the downswing, when A is low. This is easily seen in the bottom panel of Figure 8, which shows the ratio between the constrained efficient leverage and the laissez faire leverage. This shows clearly that the ratio is countercyclical. Although Figure 8 is calculated for a particular sequence of shocks, the pattern is in fact quite general.

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[12]Nuño and Thomas (2017), in an analysis of the cyclical fluctuations of the leverage ratio of US financial intermediaries in the post-war period, find that leverage has been positively correlated with assets and GDP.

The table below shows the conditional expected value of the same ratio, calculated by averaging over the entire ergodic set, conditioning on the value of the productivity shock A.

A	z^{CE}/z^{LF}
1.3169	.9527
1.1476	.9680
1.000	.9808
0.8714	.9907
0.7593	.9978

In the lowest two states, the laissez faire leverage is almost the same, on average, as the constrained efficient leverage, but it is 2% per cent higher than the constrained efficient leverage in the middle state, more than 3% higher in the second highest state, and about 5% higher in the highest state. In this *relative* sense, constrained efficiency prescribes that leverage be reduced more in the upswing of the cycle than in the downswing.

Figure 9 shows simulations of the laissez faire and regulated equilibria for the same sequence of productivity shocks. Obviously, the two equilibria with constant maximum leverage $\bar{z} = 0.55$ and $\bar{z} = 0.53$ cannot show the same fluctuations in leverage as the other two. In fact the constraints are binding for the three highest states, so it is only in the lowest two states that there is any variation in leverage. The contingent regulated equilibrium, on the other hand, exhibits very similar leverage and consumption to the laissez faire (and therefore, the constrained efficient) case. Recall that the limits on leverage are assumed to be functions of the state. More precisely, $\bar{z}(A)$ is set equal to the average constrained efficient leverage in state A, where the average is computed over the ergodic set. This suggests that the state-dependent leverage constraint may be a good approximation to the constrained efficient policy. This conjecture is given support by the correlation analysis in Table 1. The correlation between the state A and the leverage ratio z in the constrained efficient allocation, calculated over the ergodic set, is 1.00. This suggests that there is relatively little variation in z within a given state, that is, the optimal leverage is approximately a function of A. The correlation between A and z is also 1.00 in the equilibrium with contingent regulation, which suggests that the constraint $z(A)$ is binding almost all the time. More surprisingly, the correlation between A and z in the laissez faire equilibrium is also very high, 0.99. When regulation takes the form of a fixed limit $\bar{z} = 0.55$, on the other hand, the correlation between A

and z is much lower, 0.76 (and even more so if such limit is $\bar{z} = 0.53$). Of course, there is no correlation between A and z when the constraint is binding. It is only in those states where $z < \bar{z}$ that there can be any correlation between A and z. So the correlation that we do observe is explained by the variation in unregulated leverage.

6.4   Model dynamics

The dynamic properties of the model are summarized by the impulse response functions in Figures 11–14. These figures show the *average* response of the endogenous variables to a one standard deviation change in the productivity shock A_{t+1}. It is important to bear in mind two facts when interpreting the impulse response functions. First, the value of the productivity shock changes for only one period, but the responses are determined by the policy functions that anticipate the shock's serial correlation. Thus, an increase in productivity will give rise to expectation of continued high productivity in the near future and that will influence current decisions about consumption and investment. After the initial shock, the dynamics are driven solely by the predetermined values of the capital stock k and leverage z. The initial increase in the productivity shock has no effect on subsequent values of the productivity shock. Second, when the productivity shock changes, the value of leverage z has already been chosen. The predetermined leverage has asymmetric effects depending on whether the productivity A increases or decreases. If productivity increases, consumers may want to consume more, but find themselves constrained by the predetermined level of deposits. If the shock decreases and consumers want to consume less, there is nothing to stop them from reducing consumption. This asymmetry has implications for investment. When consumption is constrained after an increase in the productivity shock, the extra output must be invested.

We can get a better sense of the response of investment and consumption to productivity shocks from Figure 10. The top panel shows us the level of consumption as function of the capital stock following a productivity increase (solid lines) and decrease (dotted lines), for the constrained efficient allocation and the laissez faire equilibrium. In the constrained efficient case, the difference in consumption is small. The difference is somewhat larger in the laissez faire case, but it is small compared to the impact on investment, shown in the lower panel, where we see, in both cases, investment following a productivity increase is much higher than investment following a productivity

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decrease. This shows that the impact of a change in productivity is mainly absorbed by changes in investment. When productivity rises, consumption is constrained by deposits so increased output flows into investment. When productivity falls, on the other hand, deposits do not prevent consumption from falling but it is either maintained (in the constrained efficient case) or slightly reduced (in the laissez faire equilibrium). Again, investment absorbs most of the impact.

Figure 11 shows the effect of an increase in A on the laissez faire equilibrium and constrained efficient allocation in the ergodic set. The leverage at $t = 1$ is determined at $t = 0$, so the spike in leverage at $t = 1$ reflects the response to the increase in A and, more importantly, the expectation of a higher than normal value of A in the future. At values of $t > 1$, it is clear that both expectations and leverage have returned to normal.

Consumption initially rises only slightly, in the constrained efficient allocation, because it is constrained by the level of deposits chosen at $t = 0$. The spike in consumption occurs at $t = 2$, after deposits have been increased at $t = 1$. By contrast, there is no spike in consumption in the laissez faire equilibrium and the path of consumption is much smoother. In both cases, consumption remains somewhat elevated for a long time because the capital stock is elevated.

Investment, unlike consumption, rises as soon as the productivity shock increases at $t = 1$. Consumption is constrained by deposits, so all the increased output goes into investment. As soon as the productivity returns to "normal," investment falls, somewhat more sharply in the case of the constrained efficient allocation than in the case of laissez faire. Thereafter, investment returns to "normal" and the capital stock, after jumping up at $t = 2$, gradually declines as a result of depreciation.

Figure 12 shows the same information for the laissez faire and regulated equilibria. The qualitative features are similar to those in Figure 11, except that, in the regulated equilibria with non-contingent upper bounds on leverage, there is no spike in consumption. The equilibrium with state contingent regulation, on the other hand, resembles the constrained efficient allocation and exhibits a spike in leverage and consumption.

As we have already noted, there is an asymmetry between the effects of an increase and a decrease in the productivity shock. A decrease in A is expected to reduce consumption and investment, but consumption is not constrained by deposits. Nonetheless, there are significant differences between the laissez faire equilibrium and constrained

efficient allocation in both the timing and size of responses. Figure 13 shows impulse response functions for a one standard deviation decline in A for the laissez faire equilibrium and the constrained efficient allocation. In both cases, the leverage z changes only after a one period delay and then returns to normal. Consumption, however, drops sharply at $t = 1$ in the laissez faire equilibrium but, in the constrained efficient case, does not drop significantly until $t = 2$. In both cases, consumption returns to normal after $t = 2$. Investment responds to the negative shock at $t = 1$ and then rebounds at $t = 2$, after which investment returns to normal in both cases. The profile of the capital stock is different in the two cases, however, because the drop in investment at $t = 1$ is larger in the constrained efficient allocation, so the capital stock takes longer to recover.

Figure 14 shows the impulse response functions for the regulated equilibria. The two equilibria with non-contingent regulation have quite similar responses, except that the one with the tighter constraint, $\bar{z} = 0.53$, has a slight drop in leverage and consumption at $t = 1$ and a slightly weaker recovery in the capital stock following $t = 3$. The equilibrium with contingent regulation, not surprisingly, looks similar to the constrained efficient allocation.

As we explained above, a change in productivity has immediate and asymmetric effects because leverage, z, is predetermined, but this effect disappears as soon as leverage is adjusted, unless, of course, leverage cannot be adjusted because of a binding leverage constraint. This suggests that the impact of productivity shocks may be quite different in the medium term depending on the kind of regulation imposed. We find strong evidence of this in Table 1, which shows the correlation of productivity (A) with leverage (z), consumption (c), and investment (I), for the constrained efficient allocation and the laissez faire and regulated equilibria. The different cases fall into two groups. On the one hand, we have three cases where leverage is free to vary over the medium term: the constrained efficient allocation, the laissez faire equilibrium, and the contingently regulated equilibrium. The correlations across this group are very similar. Leverage is perfectly correlated with productivity and the correlations of consumption and investment with productivity are very high, though consumption is more highly correlated than investment. On the other other hand, we have the two regulated equilibria with constant leverage constraints ($\bar{z} = 0.55, 0.53$). In this group, the correlation coefficients are quite different. The correlation of leverage with productivity is much lower, of course, because leverage cannot vary when the constraint is binding. The correlation of

consumption with productivity is also lower and is very similar to the leverage correlation, suggesting that consumption is normally determined by deposits. The correlation of investment with productivity, on the other hand, is somewhat higher than what we observe in the other group and much higher than the correlations of consumption and leverage with productivity. In effect, non-contingent regulation extends the short term asymmetric effect of predetermined leverage, which we discussed above, into the medium term. As a result, investment has to absorb the impact of increases in productivity when consumption is constrained by (regulated) leverage.

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Figure 1: **No aggregate shock - Constrained efficient and laissez faire alloca-tion**. For the model with no aggregate uncertainty, we show the equilibrium allocation, (z_t, c_{t+1}, I_{t+1}) for a unit of capital stock, for the constrained efficient (black lines) and the laissez faire (blue lines) cases, against the state variable k_t.
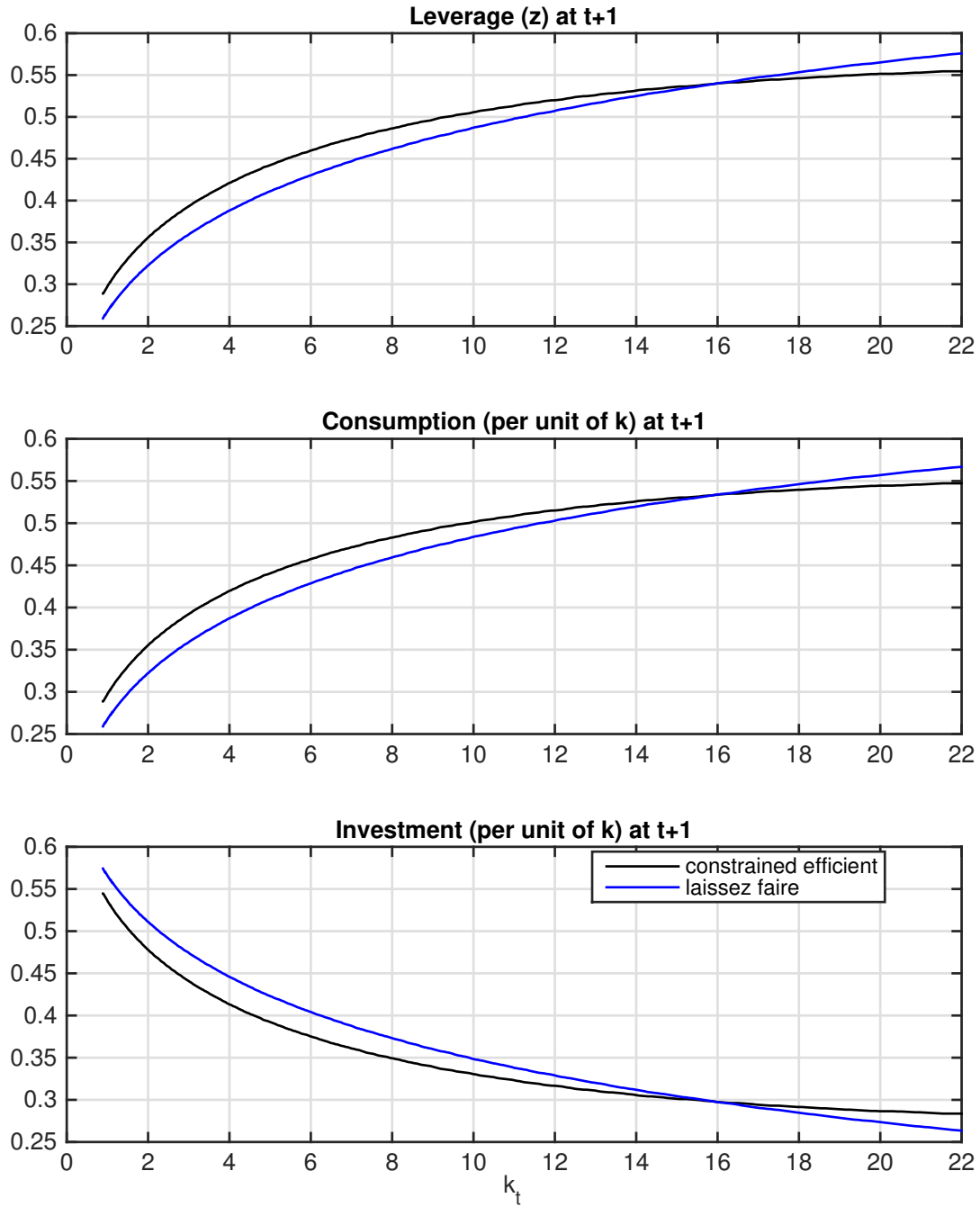
Figure 2: **No aggregate shock - Equilibrium dynamics**. For the model with no aggregate uncertainty starting from the lowest value of capital towards the steady state, we show the evolution of the equilibrium allocation, for the constrained efficient (black lines) case and the laissez faire (blue lines) case.
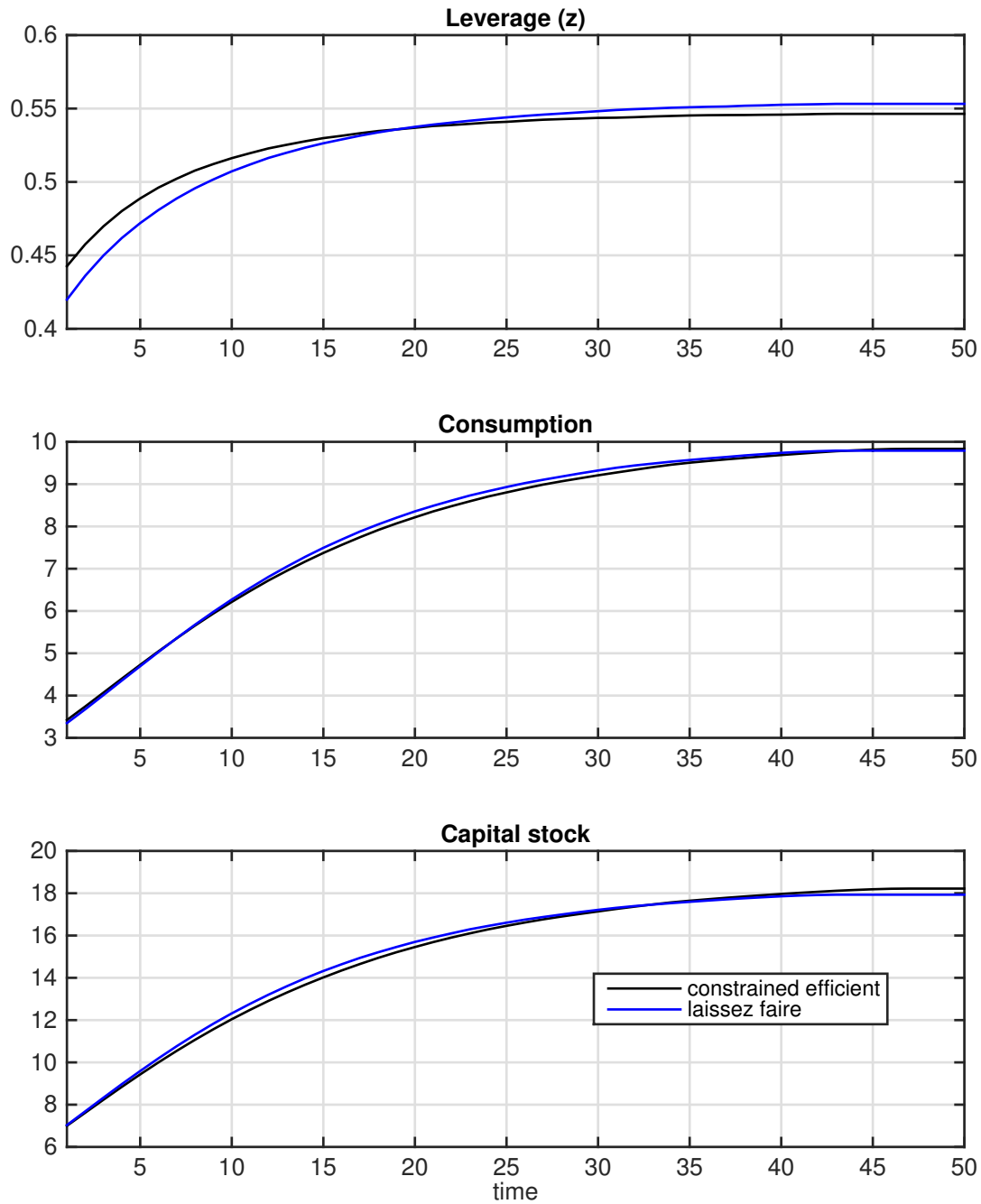
Figure 3: **No aggregate shock - Effect of regulation**. We show the state-contingent equilibrium allocation, (z_t, c_{t+1}, I_{t+1}) per unit of capital stock, against the state variable k_t. We consider the laissez faire equilibrium (blue lines), a non-contingent regulation (magenta lines), for which we assume either $\overline{z} = 0.55$ (solid magenta lines) or $\overline{z} = 0.53$ (dotted magenta lines). Finally, we consider a state-contingent regulation with upper bound $\bar{z}(k)$ (green lines).
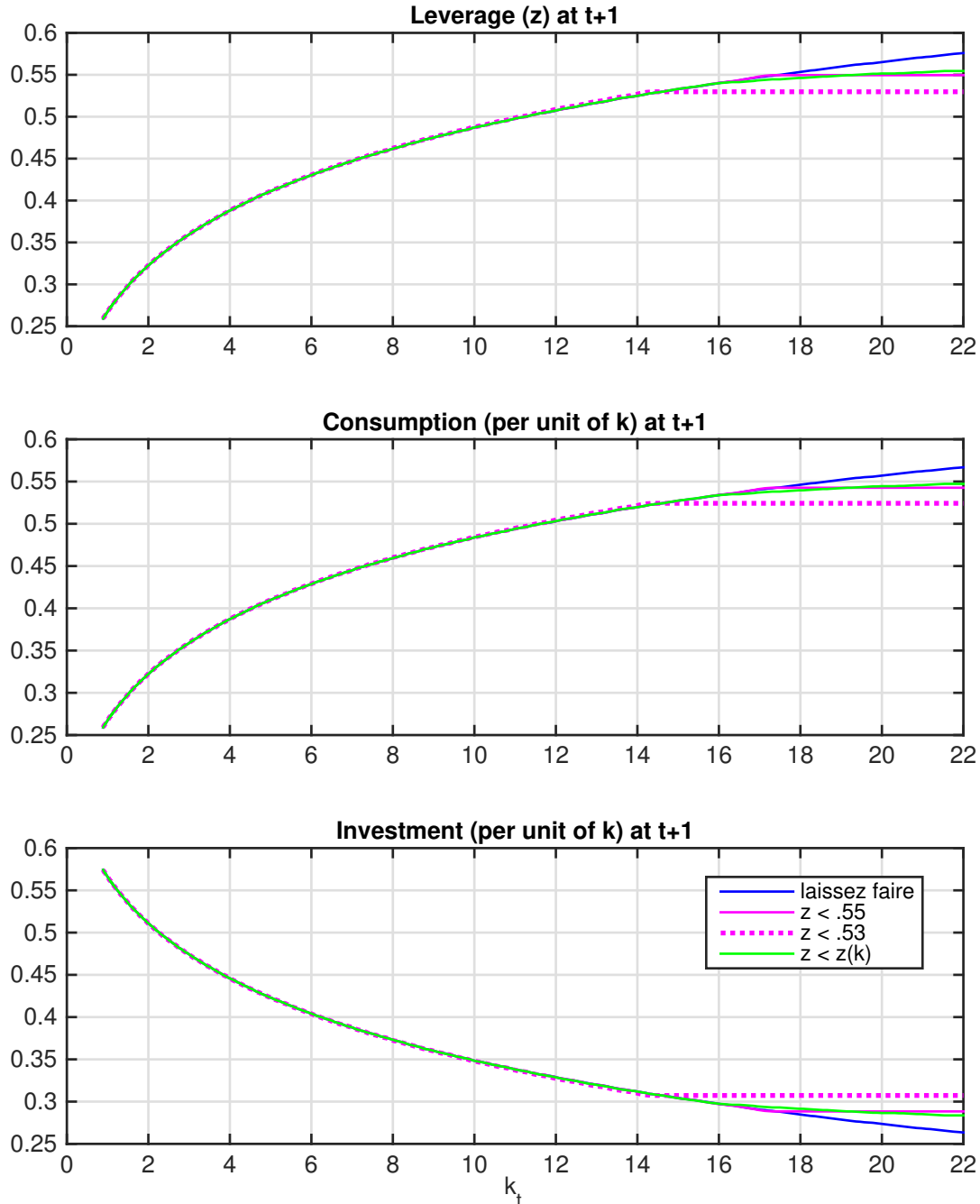
Figure 4: **No aggregate shock - Regulation dynamics**. We show the evolution of the equilibrium allocation from a model starting from the lowest value of capital towards the steady state. We consider the laissez faire (blue lines), the regulated equilibrium (magenta lines) with either $\overline{z} = 0.55$ (solid magenta lines) or $\overline{z} = 0.53$ (dotted magenta lines), and the state-contingent regulation with upper bound $\bar{z}(k)$ (green lines).
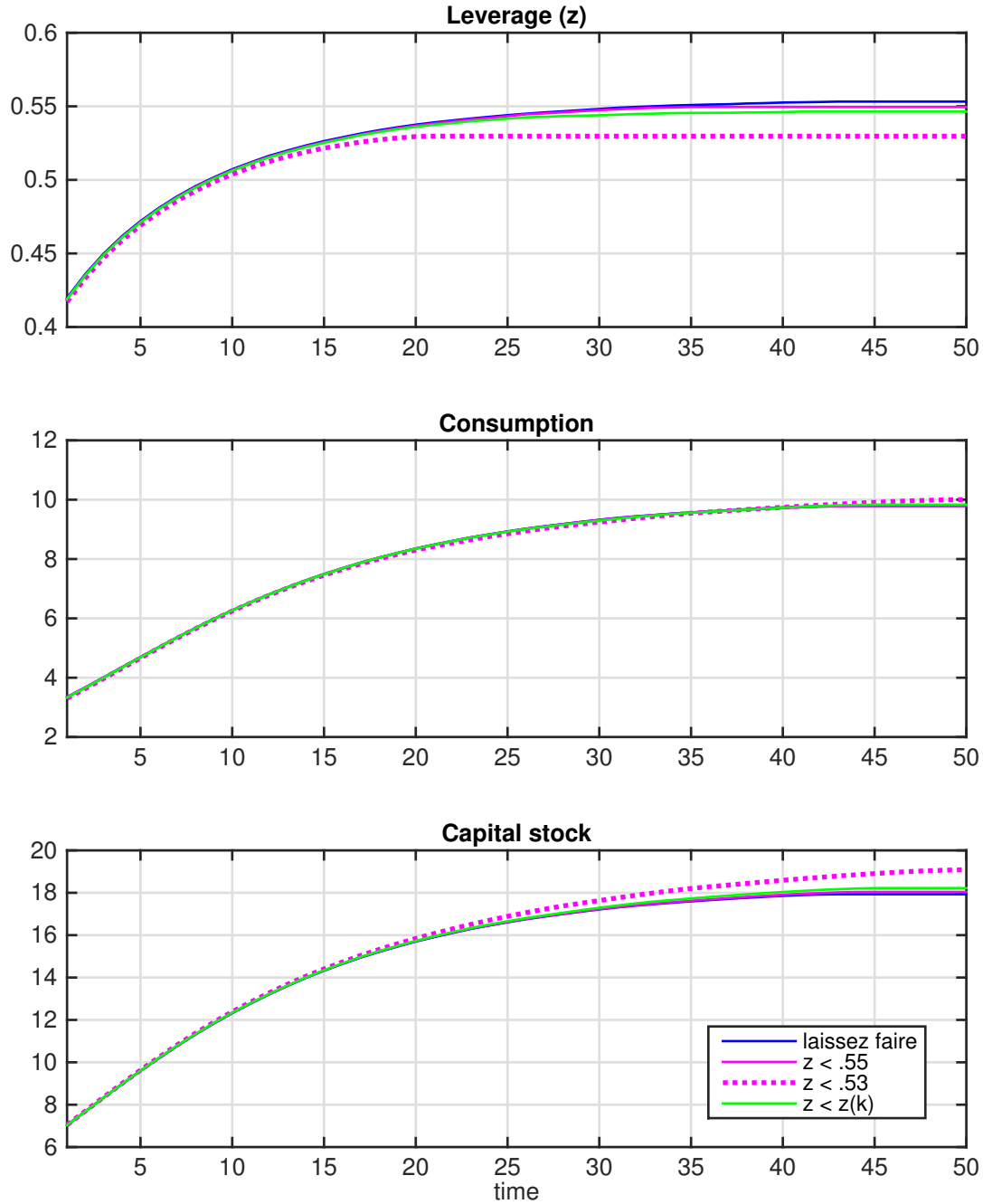
Figure 5: **No aggregate shock - Return dynamics**. For the model with no aggregate uncertainty starting from the lowest value of capital towards the steady state, we show the evolution of the return on bank's equity (solid line) and deposits (dotted line), for the laissez faire equilibrium.
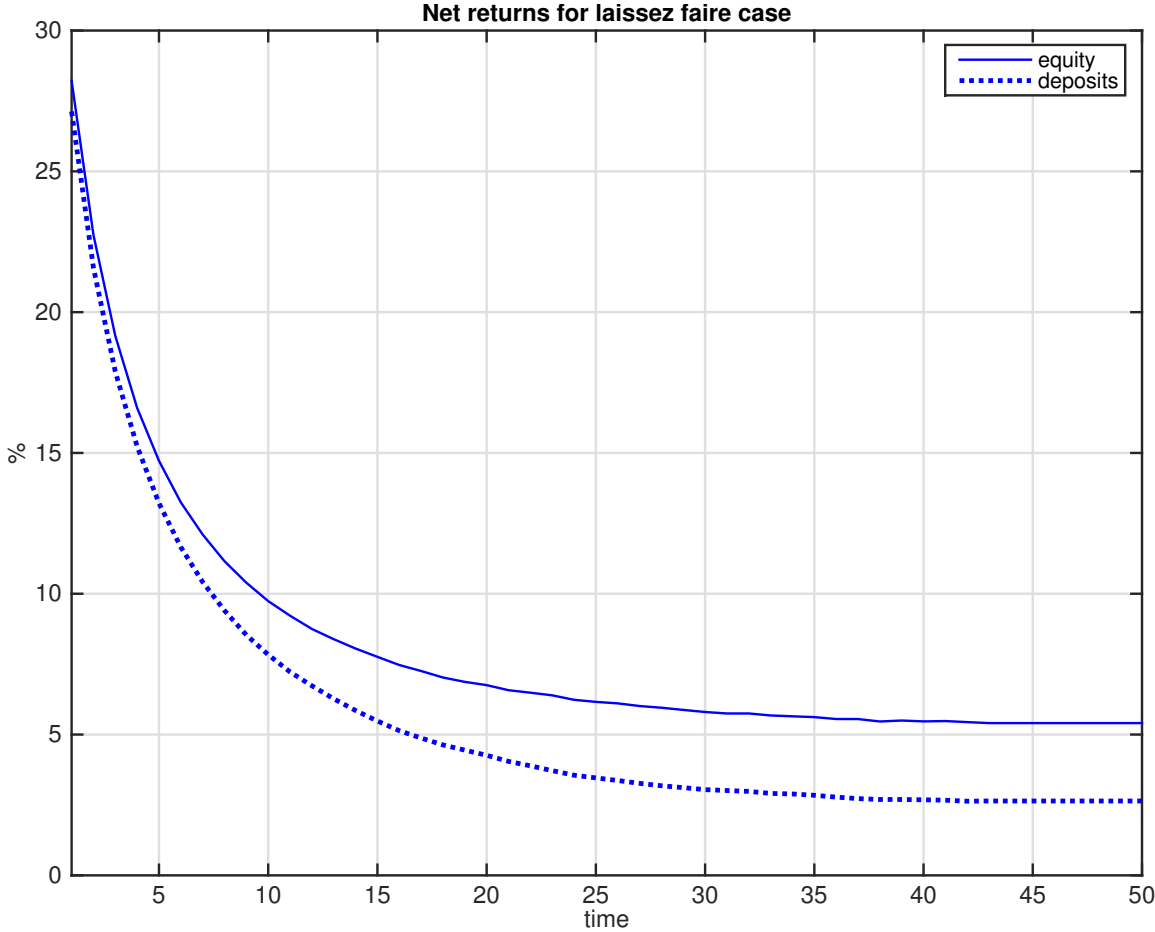
Figure 6: **No aggregate shock - Regulation and return dynamics**. We show
the evolution of the return on bank's equity (topb panel) and deposits (bottom panel)
from a model starting from the lowest value of capital towards the steady state. We
consider the laissez faire (blue lines), the regulated equilibrium (magenta lines) with
either $\overline{z} = 0.55$ (solid magenta lines) or $\overline{z} = 0.53$ (dotted magenta lines), and the state-
contingent regulation with upper bound $\bar{z}(k)$ (green lines).
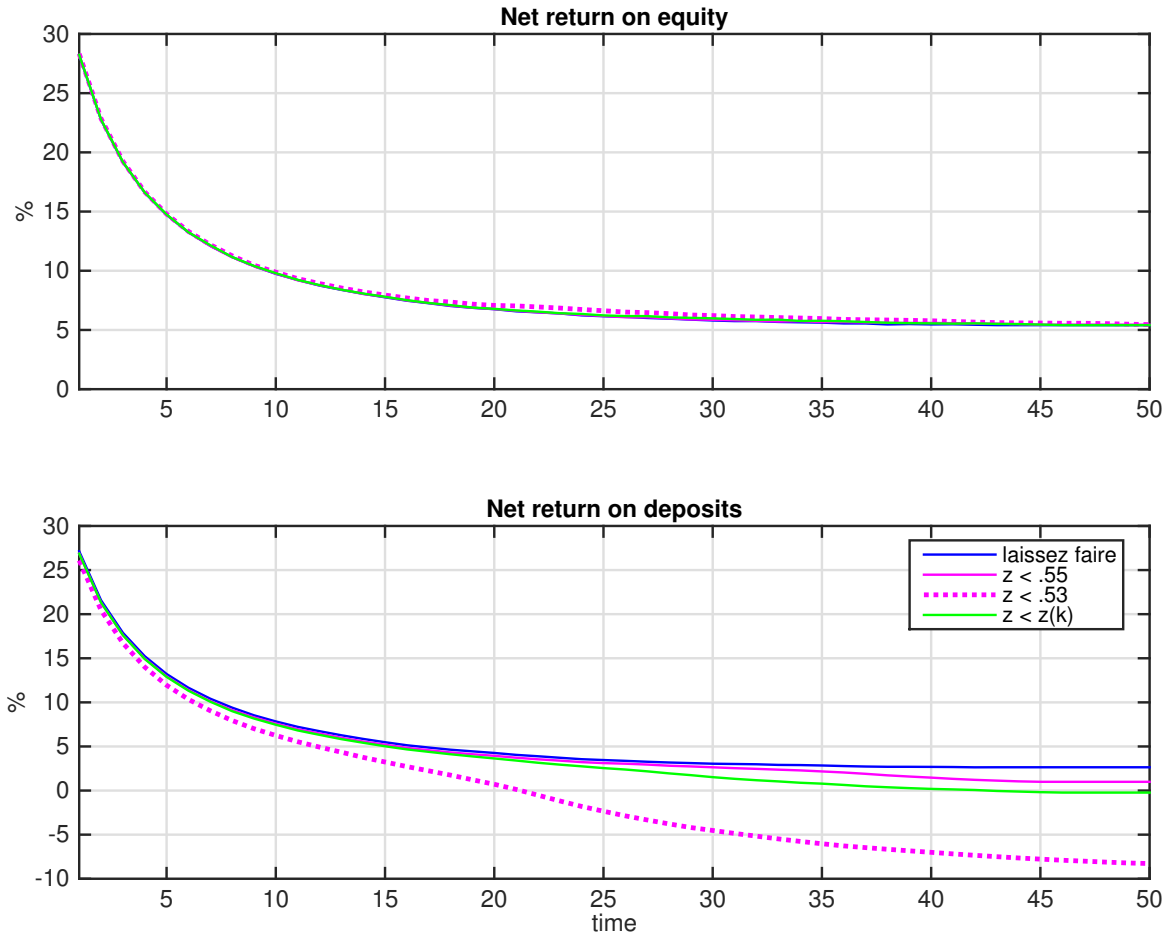
Figure 7: **No aggregate shock - dynamics of discount factor**. For the model with no aggregate uncertainty starting from the lowest value of capital towards the steady state, we show the evolution of the ratio $\psi_t / u'(c_{t+1})$. We consider four cases: laissez faire (blue lines), non-contingent regulation, in which either $\overline{z} = 0.55$ (solid magenta line) or $\overline{z} = 0.53$ (dotted magenta line). Finally, we consider the case with state-contingent regulation $z(k)$ (green lines).
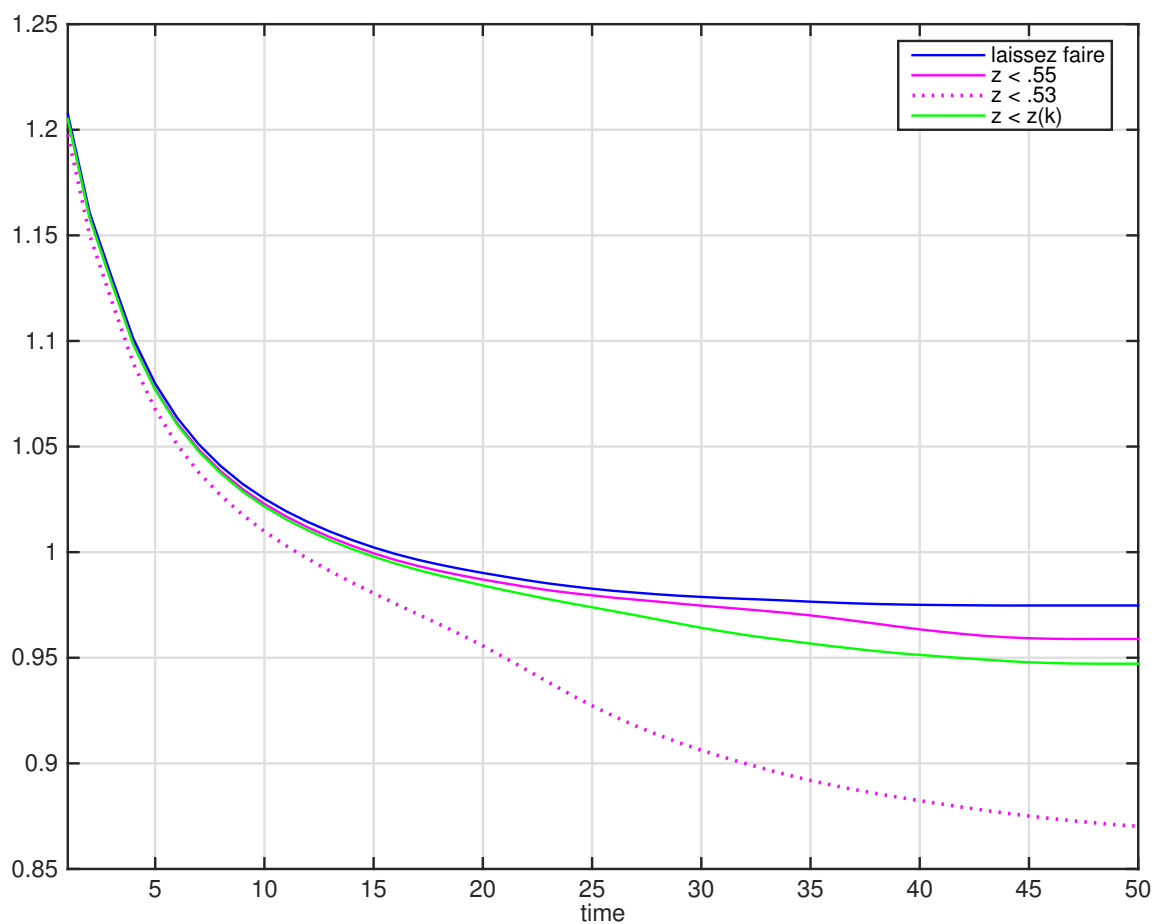
Figure 8: **Capital structure dynamics in the business cycle**. For the model with aggregate uncertainty, this figure shows the steady-state evolution of bank leverage in response to aggregate shock, A_t (upper panel) for the constrained efficient case (black line) and the laissez faire (blue line) equilibrium. In the bottom panel, we report the path of the ratio $z_t^{\text{CE}}/z_t^{\text{LF}}$. The simulation is done for the baseline parameters.
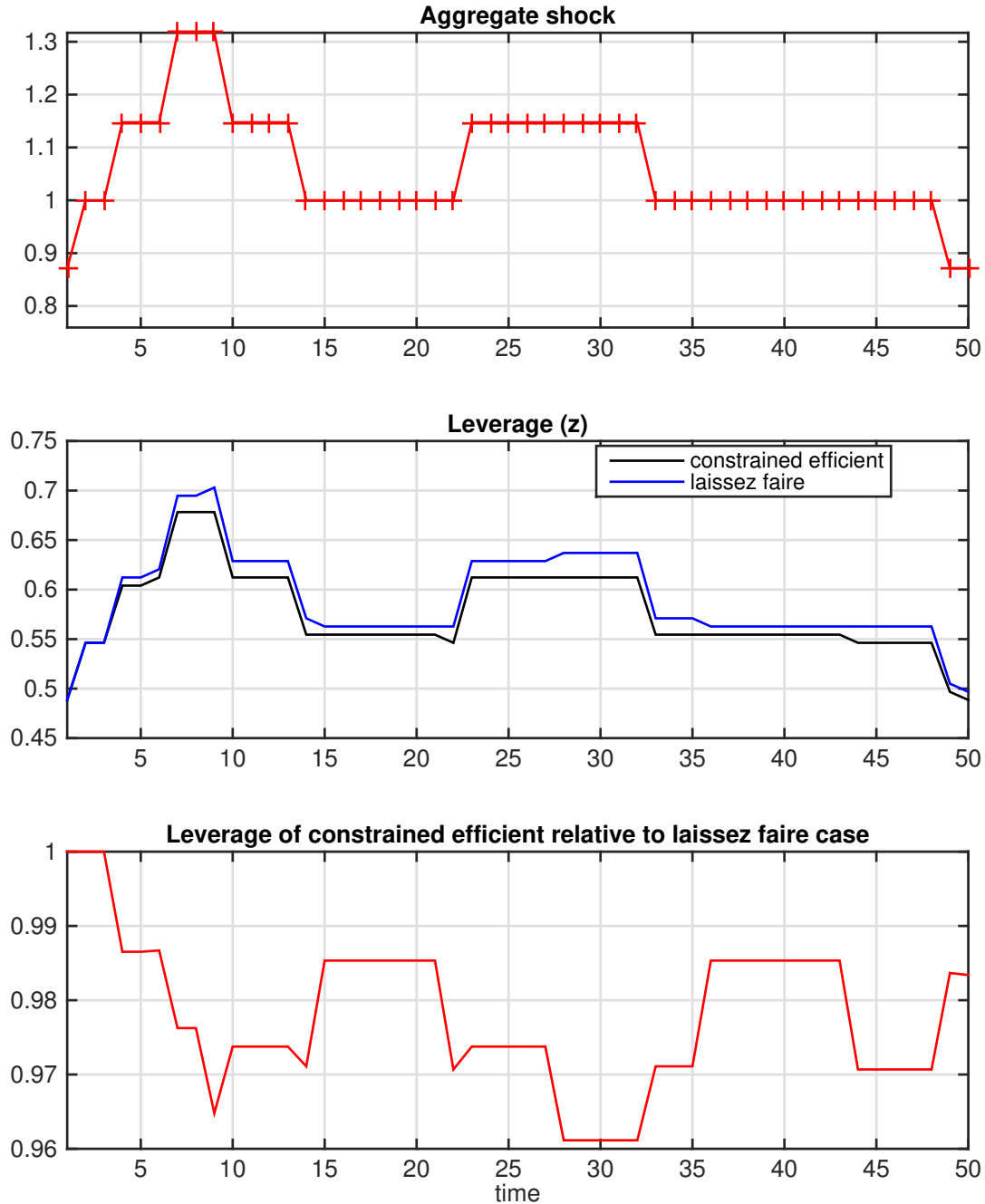
Figure 9: **Capital structure dynamics in the business cycle with regulation**.
For the model with aggregate uncertainty, this figure shows the steady-state evolution
of bank leverage of the model in response to the systematic shock, A_t, for the laissez
faire economy (blue lines), the regulated equilibrium with non-contingent restrictions on
deposits (for this case, we assume either $\overline{z} = 0.55$ – solid magenta line – or $\overline{z} = 0.53$ –
dotted magenta line), and the regulated equilibrium with state-contingent restrictions
on deposits (green line). The simulation is done for the baseline parameters.
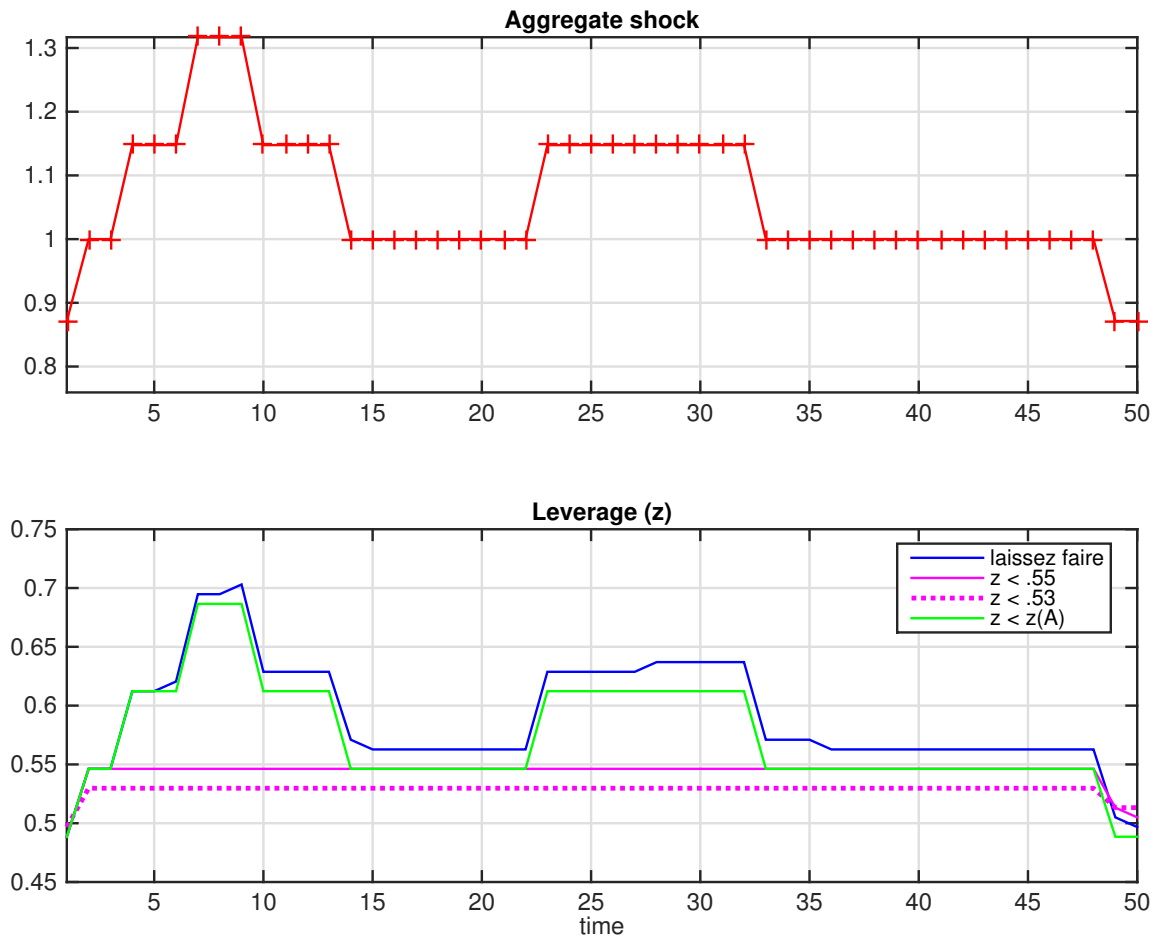
Table 1: **Correlation of allocation with business cycle.** For the model with aggregate shocks, this table shows correlations of the allocation (z, c, I) with A. Starting from a random draw of the initial capital, k_0, we simulate 100 economies for 1050 years using the law of motion of the state variables (k_t, A_t) resulting from the solution of the general equilibrium problem, and drop the first 50 years to eliminate the dependence on the initial state, (k_0, A_0). We consider the constrained efficient economy, the laissez faire economy, the regulated economy with a state-contingent constraint on leverage at $\bar{z}(A)$, and the regulated economy with a non-contingent constraint on leverage at either $\bar{z} = 0.55$ or at $\bar{z} = 0.53$. The simulation is done using the baseline parameters.

	z	c	I
constrained efficient	1.00	0.89	0.82
laissez faire	0.99	0.89	0.84
$z < \bar{z}(A)$	1.00	0.89	0.83
$z < .55$	0.76	0.75	0.87
$z < .53$	0.72	0.73	0.89

Figure 10: **Allocation response to a change of productivity.** This figure shows equilibrium consumption and investment per unit of capital stock, when the optimal (k_t, A_t)-contingent z_t has been already decided, against the state variable k_t, with A_t equal to the unconditional average of A. We assume either $A_{t+1} < A_t$ (dotted lines) or $A_{t+1} > A_t$ (solid lines). We consider the constrained efficient optimum (black lines) and the laissez faire equilibrium (blue lines).
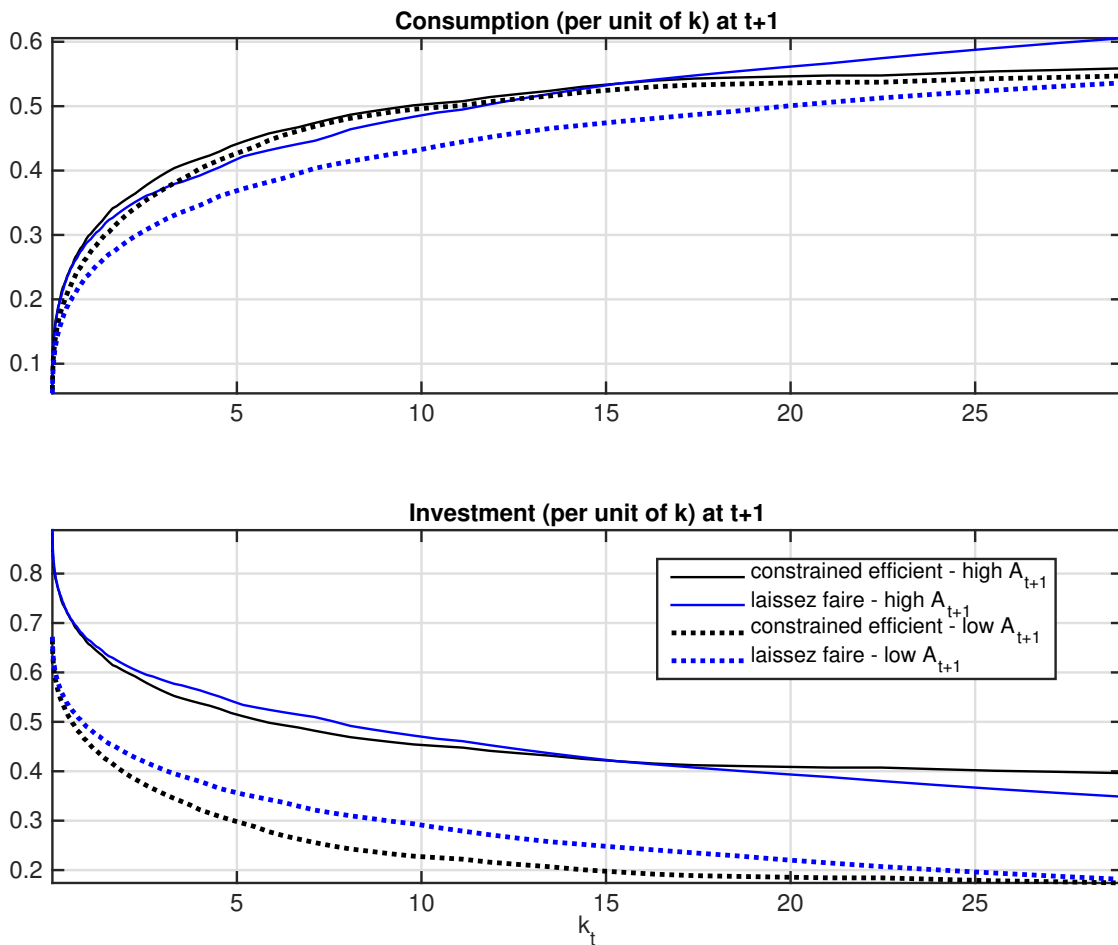
Figure 11: **Impulse responses to an upward shock on** A**.** This figure plots the impulse response of the equilibrium allocation, (z, c, I, k), in response to an upward shock of one standard deviation on A_t at time $t = 1$. There are two model specifications: the constrained efficient (black lines) and the laissez faire (blue lines). We present deviations with respect to the pre-shock steady-state level of the variable.
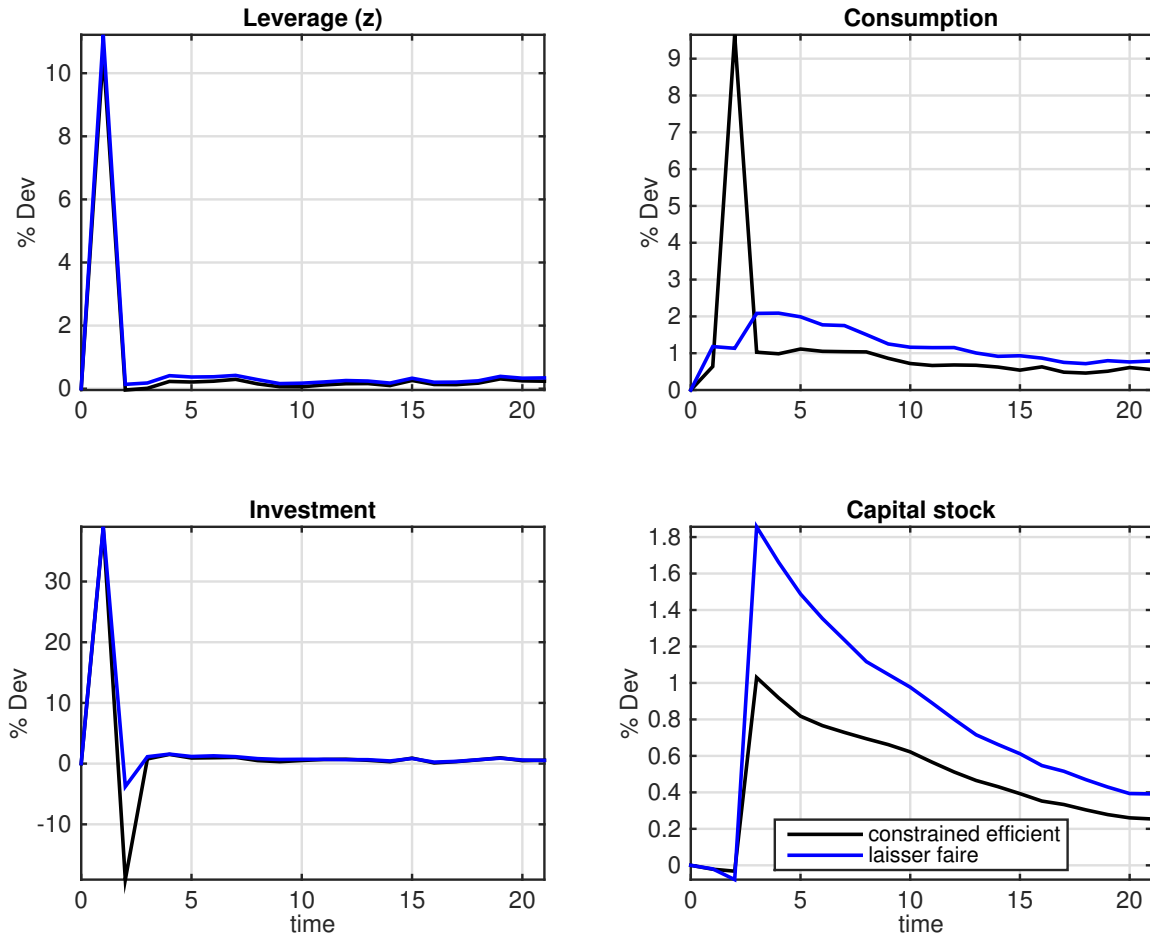
Figure 12: **Impulse responses to an upward shock on A - regulated economies.**
This figure plots the impulse response of the equilibrium allocation, (z, c, I, k), in response to an upward shock of one standard deviation on A_t at time $t = 1$. There are three model specifications: the laissez faire (blue lines), the regulated equilibrium with non-contingent restrictions on deposits (for this case, we assume either $\bar{z} = 0.55$ – solid magenta line – or $\bar{z} = 0.53$ – dotted magenta line), and the regulated equilibrium with state-contingent restrictions on leverage $\bar{z}(A)$ (green line). We present deviations with respect to the pre-shock steady-state level of the variable.

Figure 13: **Impulse responses to a downward shock on** A**.** This figure plots the impulse response of the equilibrium allocation, (z, c, I, k), in response to a downward shock of one standard deviation on A_t at time $t = 1$. There are two model specifications: the constrained efficient (black lines) and the laissez faire (blue lines). We present deviations with respect to the pre-shock steady-state level of the variable.
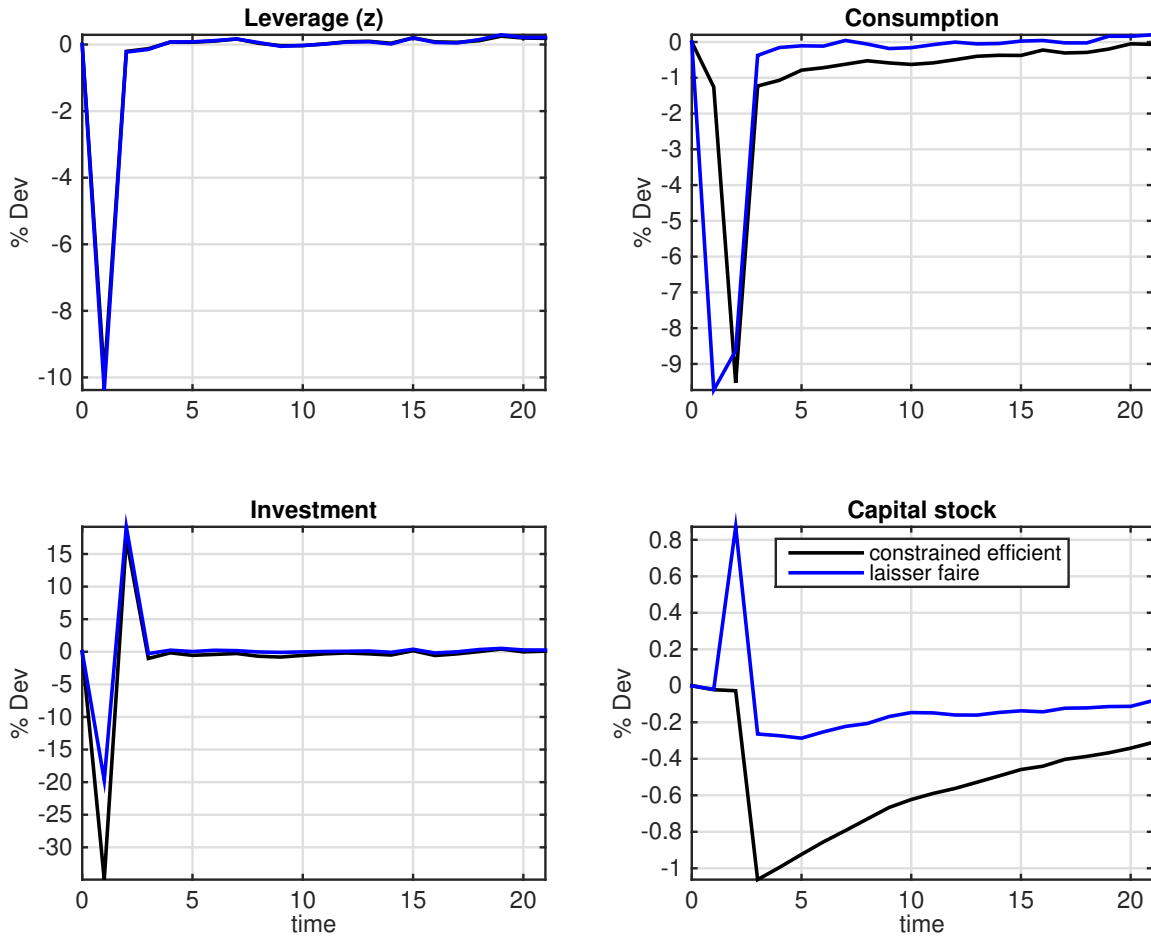
Figure 14: **Impulse responses to a downward shock on A - regulated economies.** This figure plots the impulse response of the equilibrium allocation, (z, c, I, k), in response to a downward shock of one standard deviation on A_t at time $t = 1$. There are three model specifications: the laissez faire (blue lines), the regulated equilibrium with non-contingent restrictions on deposits (for this case, we assume either $\bar{z} = 0.55$ – solid magenta line – or $\bar{z} = 0.53$ – dotted magenta line), and the regulated equilibrium with state-contingent restrictions on leverage $\bar{z}(A)$ (green line). We present deviations with respect to the pre-shock steady-state level of the variable.
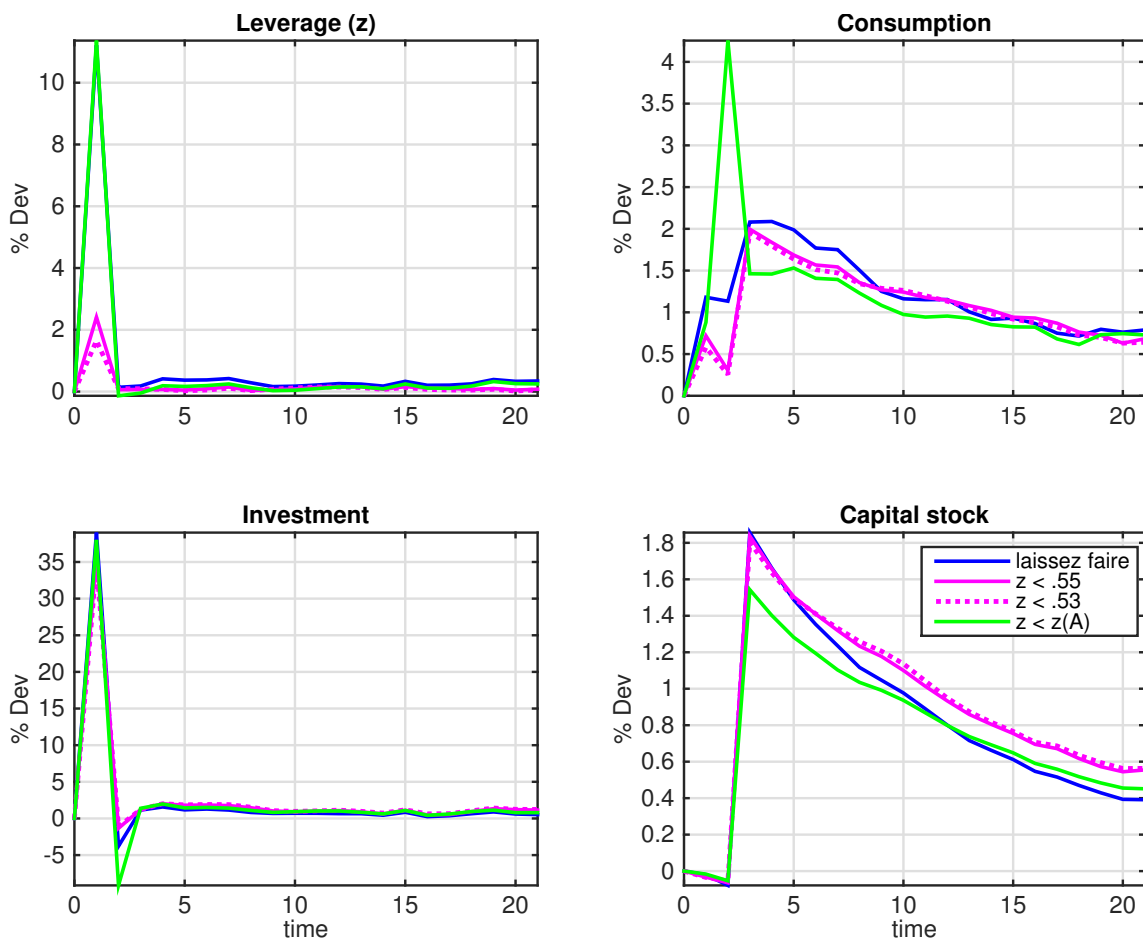
A Internet Appendix

A.1 Proof of Proposition 1

Let the total value of deposits in state A' be denoted by $D(A')$ and defined by putting

$$D(A', z) \equiv A'k(1 - \delta) \int_0^{\frac{z}{A'}} \theta dF + zk\left(1 - F\left(\frac{z}{A'}\right)\right)$$

and let total output in state A' be denoted by $Y(A', z)$ and defined by

$$Y(A', z) \equiv A'k\left(\int_0^Z \theta dF - \delta \int_0^{\frac{z}{A'}} \theta dF\right).$$

The constraint (5) will always hold as an equality at the optimum, so we can use the constraint to solve for $I(A')$ as

$$
\begin{aligned}
I(A') &\equiv A'k\int \theta dF - A'k\int^{\frac{z}{A'}} \delta\theta dF - c(A') \\
&= Y(A', z) - c(A')
\end{aligned}
$$

for each value of A'. Then the constraints (4) and (6) can be written as

$$c(A') - D(A', z) \leq 0, \text{ for each } A' \in \mathcal{A} \tag{15}$$

$$k(A') - (1 - \gamma)k - \varphi(Y(A', z) - c(A')) \leq 0, \text{ for each } A' \in \mathcal{A}. \tag{16}$$

The Lagrange multipliers corresponding to (15) and (16) are denoted by $\ell_1(A')$ and $\ell_2(A')$, respectively. The Lagrangean for the dynamic program can be written in canonical form as follows:

$$
\begin{aligned}
\mathcal{L}(\mathbf{c}, \mathbf{k}, z) = \\
\sum_A \{ &-\beta[u(c(A')) + V(k(A'), A')] + \ell_1(A')[c(A') - D(A', z)] + \\
&\ell_2(A')[k(A') - (1 - \gamma)k - \varphi(Y(A', z) - c(A'))]\} p(A'|A).
\end{aligned}
$$

The first-order condition for $c(A')$ is

$$-\beta u'(c(A')) + \ell_1(A') + \ell_2(A')\varphi'(I(A')) = 0, \ \forall A' \in \mathcal{A},$$

and the first order condition for $k(A')$

$$-\beta \frac{\partial}{\partial k} V(k(A'), A') + \ell_2(A') = 0, \ \forall A' \in \mathcal{A},$$

which establishes equations (7) and (8) in Proposition 1.

The first order condition for z is

$$\sum_{A'} \ell_0(A') \left[-\frac{\partial}{\partial z} D(A', z) \right] p(A'|A) +$$

$$\sum_{A'} \ell_1(A') \left[-\varphi'(Y(A', z) - c(A')) \right] \frac{\partial}{\partial z} Y(A', z) p(A'|A) = 0.$$

From the definitions of $D(A', z)$ and $Y(A', z)$, direct calculation shows that

$$
\begin{aligned}
\frac{\partial}{\partial z} D(A', z) &= A'k(1-\delta)\left(\frac{z}{A'}\right)F'\left(\frac{z}{A'}\right)\frac{1}{A'} + k\left(1 - F\left(\frac{z}{A'}\right)\right) - zkF'\left(\frac{z}{A'}\right)\frac{1}{A'} \\
&= -\delta k\left(\frac{z}{A'}\right)F'\left(\frac{z}{A'}\right) + k\left(1 - F\left(\frac{z}{A'}\right)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial}{\partial z} Y(A', z) &= -\delta A'k\left(\frac{z}{A'}\right)F'\left(\frac{z}{A'}\right)\frac{1}{A'} \\
&= -\delta k\left(\frac{z}{A}\right)F'\left(\frac{d}{A'}\right).
\end{aligned}
$$

Substituting these expressions into the first-order condition above and dividing by k we obtain

$$\sum_{A'} \ell_1(A') \left[\frac{\delta z}{A'} F'\left(\frac{z}{A'}\right) - \left(1 - F\left(\frac{z}{A'}\right)\right) \right] p(A'|A) +$$

$$\sum_{A_{t+1}} \ell_2(A')\varphi'(I(A'))\frac{\delta z}{A'}F'\left(\frac{z}{A'}\right)p(A'|A) = 0.$$

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The first order condition for $k(A')$ implies that $\ell_1(A') = \beta\frac{\partial}{\partial k}V(k(A'),A')$ and substituting this expression into the first order condition for $c(A')$ gives us

$$\beta u'(c(A')) - \ell_1(A') = \beta\frac{\partial}{\partial k}V(k(A'),A')\varphi'(I(A')).$$

Then substituting for $\beta\frac{\partial}{\partial k}V(k(A'),A')\varphi'(I(A'))$ in the first order condition for z gives us

$$\sum_{A'}\ell_1(A')\left[\frac{\delta z}{A'}F'\left(\frac{z}{A'}\right) - \left(1 - F\left(\frac{z}{A'}\right)\right)\right]p(A'|A) +$$
$$\sum_{A'}[\beta u'(c(A')) - \ell_1(A')]\frac{\delta z}{A'}F'\left(\frac{z}{A'}\right)p(A'|A) = 0,$$

which simplifies to

$$\sum_{A'}\beta u'(c(A'))\frac{\delta z}{A'}F'\left(\frac{z}{A'}\right)p(A'|A) = \sum_{A'}\ell_1(A')\left(1 - F\left(\frac{z}{A'}\right)\right)p(A'|A).$$

This establishes equation (9) and completes the proof of the proposition.

A.2 Proof of Proposition 2

Writing the Lagrangean for the consumer's problem defined in (10)–(13), we obtain

$$\mathcal{L}(\mathbf{c}, d, e, s) = -\sum_{A'}\{\beta u(c(A')) + \beta V(s(A');k(A'),A') +$$
$$\ell_0[qdk + rek - s] + \ell_1(A')[c(A') - \lambda(A')dk] +$$
$$\ell_2(A')[c(A') + s(A') - (\lambda(A') + \mu(A'))dk - R(A')ek - \pi(A')]\}p(A'|A).$$

The first-order conditions are as follows:

$$-\beta u'\left(c\left(A'\right)\right) + \ell_1\left(A'\right) + \ell_2\left(A'\right) \leq 0$$

$$-\beta \frac{\partial}{\partial \mathbf{s}\left(A'\right)} V\left(s\left(A'\right); k\left(A'\right), A'\right) + \ell_2\left(A'\right) \leq 0$$

$$\ell_0 q - \sum_{A'} \left\{\ell_1\left(A'\right)\lambda\left(A'\right) + \ell_2\left(A'\right)\left[\lambda\left(A'\right) + \mu\left(A'\right)\right]\right\} p\left(A'|A\right) \leq 0$$

$$\ell_0 r - \sum_{A'} \ell_2\left(A'\right) R\left(A'\right) p\left(A'|A\right) \leq 0,$$

each with its own complementary slackness condition.

The optimal values of d and e are both positive, so we obtain the pricing equations

$$\ell_0 q = \sum_{A'} \left\{\ell_1\left(A'\right)\lambda\left(A'\right) + \ell_2\left(A'\right)\left[\lambda\left(A'\right) + \mu\left(A'\right)\right]\right\} p\left(A'|A\right)$$

and

$$\ell_0 r = \sum_{A'} \ell_2\left(A'\right) R\left(A'\right) p\left(A'|A\right).$$

The conclusion of the proposition follows from the fact that $\ell_0 = m_0$, $\ell_1\left(A'\right) + \ell_2\left(A'\right) = m_1\left(A'\right)$ and $\ell_2\left(A'\right) = m_2\left(A'\right)$.

A.3   Proof of Proposition 3

The revenue received by a banker cannot exceed $a_n Z$ so there is no loss of generality in assuming that the face value of deposits satisfies $0 \leq z \leq a_n Z$. The banker's objective function is continuous and must have a maximum in the interval $[0, a_n Z]$. If $z = 0$, the derivative of the objective function is

$$\sum_{A'} \left[m_1\left(A'\right) - m_2\left(A'\right)\right] p\left(A'|A\right) > 0.$$

In fact, the marginal utility of consumption in the morning, $m_1\left(A'\right)$ is equal to $\beta u'\left(0\right)$, which is infinite. So $z = 0$ cannot be an optimum. If $z = a_n Z$, the derivative is

$$\sum_{A'} m_1\left(A'\right) \left(-\frac{\delta a_n Z}{A'}\right) F'\left(\frac{a_n Z}{A'}\right) p\left(A'|A\right) < 0,$$

because $F'\left(\frac{a_n Z}{A'}\right) > 0$ if $A' < a_n$, so $z = a_n Z$ is not an optimum. The optimum must therefore satisfy $0 < z < a_n Z$. If the banker's problem

$$\max_z \sum_{A'} \left\{ m_1\left(A'\right) \left[A' \int_0^{\frac{z}{A'}} (1-\delta)\,\theta dF + z\left(1 - F\left(\frac{z}{A'}\right)\right) \right] + \right.$$
$$\left. m_2\left(A'\right) A' \int_{\frac{z}{A'}}^{Z} \left(\theta - \frac{z}{A'}\right) dF \right\} p\left(A'|A\right)$$

has an interior solution $0 < z < a_n Z$, it must satisfy the first order condition

$$\sum_{A'} \left\{ m_1\left(A'\right) \left[A'\left(1-\delta\right)\frac{z}{A'}F'\left(\frac{z}{A'}\right)\frac{1}{A'} - zF'\left(\frac{z}{A'}\right)\frac{1}{A'} + \left(1 - F\left(\frac{z}{A'}\right)\right) \right] - \right.$$
$$\left. m_2\left(A'\right)\left(1 - F\left(\frac{z}{A'}\right)\right) \right\} p\left(A'|A\right) = 0.$$

Collecting like terms, this expression can be rewritten as

$$\sum_{A'} \left\{ m_1\left(A'\right)\left(-\frac{\delta z}{A'}\right)F'\left(\frac{z}{A'}\right) + \left[m_1\left(A'\right) - m_2\left(A'\right)\right]\left(1 - F\left(\frac{z}{A'}\right)\right) \right\} p\left(A'|A\right) = 0.$$

The first-order condition uniquely determines the value of z if the summand is increasing in z.

A.4 Proof of Proposition 4

Summing the two first order conditions in Proposition 2, we obtain

$$
\begin{aligned}
m_0\left(q+r\right) &= \sum_{A'} \left\{ m_1\left(A'\right)\lambda\left(A'\right) + m_2\left(A'\right)\mu\left(A'\right) + m_2\left(A'\right)R\left(A'\right) \right\} p\left(A'|A\right) \\
&= \sum_{A'} \left\{ m_1\left(A'\right)\lambda\left(A'\right) + m_2\left(A'\right)\left[Y\left(A'\right) - \lambda\left(A'\right)\right] + m_2\left(A'\right)v\left(A'\right)\left(1-\delta\right) \right\} p\left(A'|A\right) \\
&= \sum_{A'} \beta \left\{ u'\left(c\left(A'\right)\right)\lambda\left(A'\right) + \frac{\partial}{\partial k}V\left(k\left(A'\right),A'\right)\varphi'\left(I\left(A'\right)\right)\left[\frac{Y\left(A'\right)}{k} - \lambda\left(A'\right)\right] \right. \\
&\quad \left. + \frac{\partial}{\partial k}V\left(k\left(A'\right),A'\right)\left(1-\delta\right) \right\} p\left(A'|A\right).
\end{aligned}
$$

The envelope theorem, applied to the planner's problem, tells us that

$$
\begin{aligned}
\frac{\partial}{\partial k} V\left(k, A\right) \ = \ & \sum_{A'} \beta \left\{ u'\left(c\left(A'\right)\right) \lambda\left(A'\right) + \frac{\partial}{\partial k} V\left(k\left(A'\right), A'\right) \varphi'\left(I\left(A'\right)\right) \left[\frac{Y\left(A'\right)}{k} - \lambda\left(A'\right) \right] \right. \\
& \left. + \frac{\partial}{\partial k} V\left(k\left(A'\right), A'\right)\left(1 - \delta\right) \right\} p\left(A'|A\right),
\end{aligned}
$$

or
$$
m_0\left(q + r\right) = \frac{\partial}{\partial k} V\left(k, A\right).
$$

From the first order conditions of the planner's problem in Proposition 1, we know that

$$
\begin{aligned}
\beta u'\left(c\right) - \ell_1 \ = \ & \ell_2 \varphi'\left(I\right) \\
= \ & \beta \frac{\partial}{\partial k} V\left(k, A\right) \varphi'\left(I\right) \\
= \ & \beta \frac{\partial}{\partial k} V\left(k, A\right) \frac{1}{v}.
\end{aligned}
$$

Since $\beta u'\left(c\right)$ is the marginal utility of income in the morning, $\beta u'\left(c\right) - \ell_1$ must be the marginal utility of money in the afternoon. Thus, $\beta \frac{\partial}{\partial k} V\left(k, A\right) / v$ is equal to the marginal utility of money m_0 in the afternoon and it follows that $q + r = v$.