Bank Capital, Fire Sales, and the Social Value of Deposits*

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Abstract

We describe a model in which bank deposits yield liquidity services and therefore earn a lower rate of return than bank equity. In this sense, deposits are a cheaper source of funding than equity. The bank's equilibrium capital structure is determined by a trade off between the funding advantages of deposits and the risk of costly default. Default is costly because banks assets are sold in fire sales, which transfer value to the purchasers. This transfer is a private cost for the owners of failed banks, but not a deadweight loss for society. As a result, deposits are under-used and banks' funding costs receive a subsidy from depositors. This subsidy eventually causes banks to grow too large and accumulate too many assets.

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1 Introduction

There appears to be a consensus that banks were inadequately capitalized at the time of the financial crisis of 2007-2009. Since then, regulators have tightened capital adequacy requirements, in an attempt to make the banking system more resilient. The debate about bank capital regulation has motivated a lot of research on the determinants and optimality of bank capital structure, but some questions remain unanswered. In this paper, we contribute to the welfare economics of bank capital structure by focusing on two issues that have received

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relatively little attention. One is the social value of bank liabilities as a source of liquidity services. The other is the role of fire sales as a (private) cost of default.

The starting point for any discussion of capital structure, as Admati and Hellwig (2013) have pointed out, is the classic paper of Modigliani and Miller (1958) on the irrelevance of capital structure.[1] But there is a problem applying this theory to financial institutions, in particular, banks. The Modigliani-Miller framework was intended to apply to commercial and industrial corporations. The debt issued by these corporations is a claim to a firm's cash flow. It does not circulate as means of payment or medium of exchange. Corporate debt, in other words, has no "social value." Banks, on the other hand, raise funds in the form of deposits that function as money. Similarly, shadow banks create repo, ABS, ABCP, CLO and other near substitutes for money. These liabilities offer a liquidity and convenience yield, in addition to their interest income, which bank equity does not provide. As a result, the equilibrium interest rate on bank deposits[2] will be lower than the return on equity. In that sense, deposits are a less expensive source of funds than equity.

Because deposits are less expensive than equity as a source of funding, there must be an offsetting cost. Otherwise, banks would fund themselves entirely with deposits. One disadvantage of banks' reliance on deposits is the possibility of financial distress. Financial distress gives rise to costs with very different welfare implications. First, there are *deadweight* costs of financial distress that are *internal* to the bank. These costs may include the loss of firm specific information, the destruction of customer or counterparty networks, and the misallocation of resources when assets are sold to new owners in whose hands they are less productive. Second, there are *deadweight* costs that are *external* to the bank. These costs are imposed on other banks, firms, and individuals, and are not internalized by the bank. Finally, there are fire sales, in which assets are sold off for less than their fundamental value because markets are illiquid.[3] Fire sales are different from the other two categories because the sale of assets at a "loss" represents a private cost for the seller and a transfer of value to the buyer of the assets. It is not, therefore, a true social cost.[4] Internal deadweight costs

[1] The Modigliani-Miller Theorem states that, under certain conditions, a firm's market value is independent of its capital structure. As a corollary, the firm's cost of funding does not depend on the amount of debt and equity in its capital structure. In this sense, the cost of equity funding cannot be "expensive." In fact, to the extent that an increase in bank capital (equity) reduces the risk of default, an increase in bank equity might actually reduce the cost of funding. Thus, contrary to the claims of bankers who resist increases in capital requirements, bank capital is not "expensive."

[2] In what follows, we use the term "desposits" as a short hand for all money-like bank liabilities.

[3] The term "fire sale" generally refers to a forced sale of assets by a firm in financial distress, in which the assets are sold for less than their value to the seller. There are various reasons why the price may be low(Shleifer and Vishy, 2011). For example, the assets may be worth less to the buyer than to the seller; or the buyer and seller may have asymmetric information about the assets' true values. In this paper, we use the term "fire sale" to refer to a forced sale in which assets are sold for less than their fundamental value because of financing constraints. A fire sale, in this sense, represents a loss to the seller but not a deadweight loss to society.

[4] In non-financial corporate bankruptcies, liquidated assets may have a lower fundamental value to the firms that buy them. In this case, the transfer of physical assets results in a deadweight loss. In the case of bank failures, on the other hand, financial assets such as ABS, CP, etc., should have the same fundamental

are taken into account by the firm when it chooses its capital structure and do not give rise to a distortion. External costs, on the other hand, are ignored by the bank and may lead to excessive risk taking from the point of view of social welfare. Fire sales are taken into account by the firm, but are not true social costs, so taking them into account distorts the bank's choice of capital structure. The risk of default may be too low from the point of view of social welfare. It is because of this peculiar feature that we focus on fire sales here.

In the remainder of the paper we describe a simple, dynamic, general-equilibrium model in which banks issue socially valued deposits and banks that fail may be forced to liquidate assets in a fire sale. The convenience yield of deposits is modeled by a cash-in-advance constraint, which assumes that only deposits can be used to provide consumption within a period.[5] The convenience yield of deposits can be identified with the Lagrange multiplier of the cash in advance constraint. The convenience yield is positive if and only if the cash in advance constraint is binding.

Our main results follow directly from the interaction of the fire-sale costs of default and the social value of money. The first result is that, in equilibrium, the convenience yield on deposits is positive if and only if asset markets are illiquid and default results in a fire sale. If there were no fire sales—and hence no cost of default—banks would want to fund themselves entirely with deposits. Conversely, if default were costly, but there were no convenience yield on deposits, banks would fund themselves entirely with equity. We show that neither of these outcomes is consistent with equilibrium.

The second result is that equilibrium is Pareto efficient if and only if the economy is saturated with deposits, that is, the convenience yield of deposits is zero. So inefficiency arises precisely when deposits are a cheap source of funding. The lower return on deposits acts like a tax on consumption. At the margin, consumers reduce consumption and increasing savings. Since deposits are earmarked for consumption, saving takes the form of equity purchases. Investment is financed by retained earnings on equity, so an increase in equity leads to increased investment.

The third result is that, when deposits are cheap and equilibrium is Pareto inefficient, banks accumulate too many assets and the banking sector becomes too large in the long run. This follows directly from the low cost of debt funding that distorts economic agents decisions about savings and investment.

The final result is that inefficiency does *not* entail banks overleveraging. On the contrary, the level of bank deposits is *too low*. A rational banker perceives default as an avoidable cost: increasing leverage increases the probability of being forced to sel assets in a fire sale and decreases the probability of making capital gains from buying assets at fire sale prices. This perception distorts his decision. At the margin, he is, in fact, using too little debt from the point of view of social welfare.

Stein (2012) has some similarities with the present paper. In particular, he assumes that the safe component of deposits bears a liquidity premium or convenience yield and that fire

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value to other banks.

[5] The returns from equity must be converted into deposits before they can be consumed and this conversion takes time, which is costly to consumers.

sales result in transfers of value but are not socially costly. Like us, he finds that the liquidity premium on (safe) deposits gives rise to overinvestment, but the explanation is different. In Stein's model, overinvestment occurs because banks need more collateral in order to increase the safe component of deposits. In contrast, in our model the level of deposits is too low from the point of view of social welfare,[6] rather than too high. The models are very different, of course. Stein (2012) considers a static model in which bank assets are subject to a perfectly correlated shock and consumers are assumed to to get additional utility from safe deposits. In this paper, we consider an infinite horizon model, in which banks are subject to purely idiosyncratic shocks (there is no aggregate uncertainty) and the social value of deposits is derived from the standard utility and asset pricing theory.

Our objective is to use a simple, general equilibrium model to illustrate some aspects of the welfare economics of bank regulation. We focus on fire sales, while ignoring other, potentially important, costs. In some cases, our qualitative conclusions would be the same if deadweight costs were included. If the deadweight costs are internalized by the bank, they will be taken into account when the banker balances the cost of default against the funding advantages of deposits. The presence of deadweight costs will alter the capital structure quantitatively, but the deposit to equity ratio will still be lower than socially optimal because of the fire sale cost. On the other hand, external costs of financial distress create a second distortion that works in the opposite direction to fire sales. The relative size of these two distortions is, of course, an empirical matter. It may be that the deposit to equity ratio is too high when external costs are sufficiently high, but the effects we obtain will still be present. Practical policy recommendations will obviously require a richer model than the one presented here.

We start with a simple growth model in which two goods are produced, a capital good and a consumption good. Capital goods are used to produce consumption goods subject to constant returns to scale: k units of capital produce Ak units of consumption goods on average. The consumption goods can either be consumed or transformed into capital goods using a neoclassical production function with decreasing returns to scale.

Only bankers can manage capital goods (assets). Bankers have no resources of their own, so they issue debt (in the form of deposits) and equity to fund the purchase of assets. Deposits are special because only deposits can be used to purchase consumption goods. So, even though equity pays a higher return, consumers will hold deposits in order to consume.

There is a continuum of individual banks whose investment returns are subject to idiosyncratic shocks. If bank i invests in k units of capital at date t, it will produce $\theta_i Ak$ units of consumption at date $t+1$, where θ_i is a random variable with mean one. Because these shocks are i.i.d. across banks, there is no aggregate uncertainty. In the absence of markets to insure against these idiosyncratic risks, banks may find they have too little cash to honor the demands of their depositors. In that case, they are forced to default and liquidate their

[6] If we compare the laissez faire economy with a socially optimal economy starting from the same initial conditions, the capital stock in the laissez faire economy will be higher and, as a result, the absolute amount of deposits may eventually be higher. But, given the same capital stock, the level of deposits in the laissez faire economy will be too low.

assets (capital goods). Limited liquidity in the asset market leads to fire sales that impose a private cost on failing banks that are forced to liquidate assets. At the same time, fire sales offer the opportunity for solvent banks to purchase assets at fire-sale prices and make a capital gain. Banks choose a capital structure that maximizes the market value of the securities (deposits and equity) they issue.

Consumers choose a portfolio of shares and deposits to maximize their individual utility. This involves a tradeoff between the higher return they get from holding equity and the convenience yield of being able to use deposits for consumption.

The rest of the paper is organized as follows. In Section 2 we describe a general equilibrium model of a banking economy and define a competitive equilibrium. In Section 3, we show that the optimal bank capital regulation policy can implement the first best. We derive necessary and sufficient conditions for equilibrium to be efficient in Section 4. In Section 5, we show that inefficiency takes the form of overaccumulation. In Section 6 we summarize our findings and discuss some topics for further research.

1.1 Related Literature

Following on the seminal paper of Modigliani and Miller (1958), a large literature has grown up investigating the role of various factors, such as, taxes, bankruptcy, term structure, seniority and incentive problems, on the choice of firm capital structure. A (non-representative) sample of this vast literature includes Brennan and Schwartz (1978), Hackbarth and Mauer (1979), Barnea, Haugen and Senbet (1981), Kim (1982), Titman (1984), Dammon and Green (1987), Titman and Wessels (1988), Leland and Toft (1996), and Bradley, Jarrell, and Kim (2011).

Our paper is related to the literature on bank capital structure.[7] Various papers have shown that deposits are typically the optimal form of funding for banks, since they protect uninformed agents and create liquidity (Gorton and Pennacchi 1990), provide liquidity insurance (Diamond and Dybvig 1983), allow banks to act as delegated monitors (Diamond 1984), or discipline bankers and provide incentive benefits (Calomiris and Kahn 1991; Flannery 1994; Diamond and Rajan 2001).[8] There has been a long-lasting debate about the treatment of deposits in calculating output generated by banks. On the one hand, deposits are an input into the generation of earning assets. On the other hand, they provide liquidity, safekeeping, and payments services to depositors.[9] Hence, deposits are simultaneously an input and an output, which puts them in a special position compared to debt in the traditional Modigliani-Miller framework. Our paper exploits this feature of deposits, where they

[7] Berger, Herring and Szego (1995) illustrate historical trends in bank capital in the US. Hanson, Shleifer, Stein and Vishny (2015) report that deposits averaged 80% of total assets of US commercial banks over the period 1896-2012. Flannery and Rangan (2008) document the variation in US banks' capital ratios in the last decade.

[8] Also, see Holmström and Tirole (1998, 2011) and Gorton (2010) for the role banks in liquidity creation.

[9] Sealey and Lindley (1977) treat deposits as inputs and Berger, Hanweck and Humphrey (1987) treat them as outputs, to cite a few. Also, see Fixler and Zieschang (1992). For an excellent review, see Berger and Humphrey (1997).

are needed for payments and consumption due to the cash-in-advance constraint.

Gale (2004) extends the Diamond-Dybvig model to include bank capital that provides additional risk sharing between risk-neutral investors (equity holders) and risk-averse depositors. Diamond and Rajan (2000) build a model, where fragility of deposits are essential for banks to create liquidity. Bank capital can reduce financial distress but also reduces liquidity creation. Hence, the optimal bank capital structure trades off liquidity creation and the cost of financial distress. Allen, Carletti and Marquez (2015) build a model where the markets for deposits and equity are segmented and capital can help prevent costly bankruptcy. They show that, in equilibrium, equity capital earns a higher return than investing directly in the risky asset, which in turn has a higher expected return than deposits, related to the earlier findings on the cost of capital provided by Myers and Majluf (1984). DeAngelo and Stulz (2015) build a model, where there is a market premium for safe/liquid debt.[10] Banks with risky assets use risk management to maximize their capacity to have such debt in their capital structure and have higher leverage than non-financial firms. Flannery (2012) analyzes bank capital structure decisions when liquidity is priced at a premium. Sundaresan and Wang (2016) develop a dynamic continuous-time model of optimal bank liability structure that incorporates the liquidity services of deposits, deposit insurance, regulatory closure, and endogenous default in banks' financing decisions.

In our paper, pecuniary externalities arise from and fire sales play a key role in our welfare analysis. It is well known that pecuniary externalities have an impact on welfare in the presence of market incompleteness, information asymmetries, or other frictions (Arnott and Stiglitz 1986; Greenwald and Stiglitz 1986; Geanakoplos and Polemarchakis 1986). Pecuniary externalities and overinvestment have also been studied by Lorenzoni (2008), Gale and Gottardi (2011, 2015), Davila and Korinek (2017).

The financial and corporate sectors are merged in our model and banks are assumed to invest directly in capital goods. Gornall and Strebulaev (2015) and Gale and Gottardi (2017) develop models in which the financial is separate from the corporate sector and banks lend to firms that invest the money in real assets. These authors assume that default results in real (deadweight) costs on firms and banks and pecuniary externalities play no role.

There is a substantial empirical literature on the costs of default, for both banks and non-financial firms. The literature shows that these costs can be substantial for both banks and non-financial firms (see James, 1991; Andrade and Kaplan, 1998; Korteweg, 2010). More recent work suggests that these estimates may understate the true costs of default (Almeida and Philippon, 2007; Acharya, Bharath and Srinivasan, 2007). Deadweight costs of default take various forms, the cost of the bankruptcy process itself, the loss of firm-specific knowledge, relationships and networks, misallocation of assets to owners who cannot manage them efficiently, and costs of distress imposed on other firms that are not internalized by the defaulting bank. A smaller literature focuses on fire sales. Ellul et al. (2011) measure the effect of fire sales following downgrades of corporate debt. Regulatory requirements force insurance companies to sell assets following a bond downgrade, when the market will be

[10] Krishnamurthy and Vissing-Jorgensen (2012) provide evidence that Treasury security prices embed a liquidity premium, that is, there is a premium for safe and liquid assets.

thin because other insurance companies are major holders of corporate bonds. Meier and Servaes (2015) show that firms that buy distressed assets earn higher returns than regular acquisitions. Merrill et al. (2012) show that distressed insurance companies sold assets at lower prices than non-distressed companies. Shleifer and Vishny (2011) provide a survey of the financial and macroeconomic literature on fire sales.

A few papers, such as Gomes and Schmid (2016) and Miao and Wang (2010), build dynamic general equilibrium models and study a representative agent economy, where firms finance investments with debt and equity. The liquidation cost, in case of default, is exogenous in their setup. In that sense, our paper is close to and builds on the setup developed in Gale and Gottardi (2015), where the liquidation price is endogenously determined in equilibrium. Our paper is, therefore, related to the literature on fire sales (Williamson 1988; Shleifer and Vishny 1992) and cash-in-the-market pricing (Allen and Gale 1994, 1998), where the price of assets are determined by the available liquidity in the market. Several papers provide empirical evidence on fire sales. Pulvino (1998) analyzes the prices of used airplanes, where he compares the prices of planes sold by financially distressed airlines to those sold by the airlines that were not distressed, controlling for the characteristics of the planes. He finds that used planes sold by distressed airlines bring 10 to 20 percent lower prices than planes sold by undistressed airlines.[11] Other studies provide similar effects of fire sales. For example, Campbell, Giglio, and Pathak (2011) report in a study of forced home sales that foreclosure discounts are on average 27 percent of the value of the house. Meier and Servaes (2015) show that firms that buy distressed assets earn higher returns than regular acquisitions.

Financial assets also suffer from fire sales. Coval and Stafford (2007) provide evidence from the mutual fund industry. They show that funds experiencing large outflows decrease existing positions, which creates price pressure in the securities held in common by distressed funds, and investors who trade against constrained mutual funds earn significant returns. Ellul, Jotikasthira, and Lundblad (2011) investigate fire sales of downgraded corporate bonds by insurance companies. They show that insurance companies constrained by regulation are more likely to sell downgraded bonds leading to price declines in such bonds, where the price effects are larger during periods when the insurance industry is distressed and other potential buyers' capital is scarce. Merrill et al. (2012) show that distressed insurance companies sold assets at lower prices than non-distressed companies. Mitchell, Pedersen and Pulvino (2007) investigate the convertible bond market in 2005 when convertible hedge funds faced large redemptions resulting in binding capital constraints for many funds and massive bond sales by such funds. These sales reduced prices of convertibles relative to fundamental values, especially around redemptions. Allen and Gale (2004a, 2004b) provide models of fire sales

[11] Acharya, Bharath and Srinivasan (2007) provide similar evidence for fire sales using data on defaulted firms in the United States over the period 1982–1999 (see also Berger, Ofek, and Swary 1996; Andrade and Kaplan 1998; Stromberg 2000; and Korteweg 2010). Other research shows that firms try to avoid fire sales of assets in illiquid markets such as Asquith, Gertner, and Scharfstein (1994) who find that, when industry conditions are bad, a debt work-out is more likely than a liquidation (also see Schligemann, Stulz, and Walkling (2002) and Almeida, Campello, and Hackbarth (2011)).

and their impact on bank portfolios. Shleifer and Vishny (2011) provide a survey of the financial and macroeconomic literature on fire sales.

Regarding the US financial crisis in 1907, Cleveland and Huertas (1985) write (page 52): National City Bank again emerged from the panic a larger and stronger institution. At the start, National City had higher reserve and capital ratios than its competitors, and during the panic it gained in deposits and loans relative to its competitors. Stillman (President) had anticipated and planned for this result. In response to Vanderlip's (Vice President) complaint in early 1907 that National City's low leverage and high reserve ratio was depressing profitability, Stillman replied: "I have felt for sometime that the next panic and low interest rates following would straighten out good many things that have of late years crept into banking. What impresses me as most important is to go into next Autumn (usually a time of financial stringency) ridiculously strong and liquid, and now is the time to begin and shape for it. If by able and judicious management we have money to help our dealers when trust companies have suspended, we will have all the business we want for many years."

Finally, our paper is related to the literature on the optimal size of the financial sector, which attracted much attention after the crisis.[12]. Prior to the recent crisis, the empirical literature has mainly documented a positive relationship between financial sector deepening and economic development.[13] Rajan and Zingales (1998) find that industrial sectors that are in need of external finance develop disproportionately faster in countries with more developed financial markets. Recent studies have found that the positive relation between financial development and economic growth holds only up to a threshold level of credit to GDP and beyond a critical level of financial development, there is no association or even a negative relation.[14] Philippon (2010) builds a theoretical model where human capital is allocated between entrepreneurial and financial careers. Cecchetti and Kharroubi (2015) examine the negative relationship between the rate of growth of the financial sector and the rate of growth of total factor productivity. They show that by disproportionately benefiting high collateral/low productivity projects, an increase in finance reduces total factor productivity growth.

2 The model

Time is discrete and indexed by $t = 0, 1 \ldots$. At each date t, there are two goods. One is a perishable consumption good; the other is a durable capital good. The consumption good is produced, subject to constant returns to scale, using capital goods as the only input. Capital goods are produced, subject to decreasing returns to scale, using the consumption good as the only input.

[12] See, for example, "Warning: too much finance is bad for the economy", *Economist*, 18 February 2015. Also, see ESRB (2014) on an analysis on the size of the banking sector in Europe.

[13] For excellent surveys, see Levine (1997) and Levine (2005).

[14] See Arcand, Berkes and Panizza (2012), Cecchetti and Kharroubi (2012) and Barajas et al. (2013), to cite a few.

The economy consists of three types of economic agents: bankers, consumers, and producers. Only bankers have the expertise to manage capital goods. The bankers have no initial endowment, so they issue equity and deposits to finance the purchase of capital goods. The revenue generated by the bankers is paid out as principal and interest to depositors and as dividends to shareholders. Free entry and competition ensure that bankers receive no remuneration and maximize the value of the firms under their control.

Consumers are the initial owners of capital goods, which they sell to the bankers in exchange for deposits and equity. The consumers invest the proceeds from the sale of capital goods in a portfolio of deposits and equity to fund future consumption. In subsequent periods, consumers manage their portfolios to provide an optimal consumption stream.

Producers use consumption goods as an input to produce capital goods. Because production is instantaneous, there is no need to raise finance for this operation. Producers use the revenue from selling their output to pay for inputs. Inputs and outputs are chosen to maximize profits in each period.

—— Insert Figure 1 here ——

Figure 1 illustrates the sequence of events in the model. The time period $[t, t+1)$ starting at date t is divided into three sub-periods, labeled A, B and C. Deposits are withdrawn in sub-period A. All consumption takes place in sub-period A and cannot exceed the level of the consumer's bank deposits. This "cash-in-advance" constraint captures the idea that one must pay for consumption goods with deposits. Payments received in sub-periods B and C can be exchanged for deposits and equity in sub-period C, but deposits created in sub-period C cannot be consumed until sub-period A of period $t+1$.

Individual banks are subject to idiosyncratic productivity shocks. As a result of a low productivity shock, a bank may have insufficient funds to meet the demand for withdrawals. In that case, it is in default. After it pays out what it can in sub-period A, it goes through the bankruptcy procedure and liquidates its capital stock in sub-period B. Banks that have sufficient funds to meet withdrawals in sub-period A retain their earnings (revenue minus withdrawals) and use the retained earnings to purchase capital goods in sub-period B. The proceeds from the sale of liquidated capital goods are divided between depositors and shareholders, with depositors being the senior claimants.

In sub-period C, consumers purchase newly issued deposits and equity and rebalance their portfolios. Banks issue deposits and equity to adjust their capital structure and to acquire new capital goods. They may also dispose of excess capital goods to newly formed banks. For simplicity, we assume all banks have the same amount of assets under management at the end of sub-period C. They will also choose the same capital structure.[15]

[15] We show that the optimal capital structure is uniquely determined, given the size (capital stock) of the bank. Because we assume constant returns to scale, the size of individual banks is immaterial.

2.1 Assumptions

All real assets are held by bankers. There is a unit mass of bankers represented by the interval $[0,1]$. Each banker $i \in [0,1]$ receives a productivity shock $\theta_{it} \in [0,\Theta]$ at date t. One unit of capital produces $\theta_{it}A$ units of the consumption good in sub-period A at date t. Let $F(\theta)$ denote the c.d.f. of the random variables $\{\theta_{it}\}$. The following assumptions are maintained throughout.

Assumption F is a continuous function with $F(0) = 0$ and $F(\Theta) = 1$; $F(\theta)$ is increasing on $[0,\Theta]$ and the hazard rate $F'(\theta)/(1 - F(\theta))$ is increasing on $[0,\Theta]$.

The assumption of an increasing hazard rate ensures that the banker's objective function is concave. We assume that the random variables $\{\theta_{it}\}$ are i.i.d. across i and t and satisfy the "law of large numbers" convention, that is,

$$\int_0^\Theta \theta_{it} di = \mathbf{E}[\theta_{it}] = 1,$$

for any i and t.

Capital is subject to depreciation. One unit of capital is reduced to $1-\delta$ units of capital after production occurs.

The technology for producing capital goods is subject to decreasing returns to scale: $I \geq 0$ units of the consumption good produce $\varphi(I)$ units of capital goods instantaneously. The profits from production of capital are distributed to consumers each period. The production function φ is assumed to have the usual neoclassical properties.[16]

There is a large number of identical and infinitely lived consumers. We normalize the mass of consumers to be one. A consumer begins life with k_0 units of capital goods at date 0 that he sells to bankers in exchange for deposits and equity. We assume there is no consumption or production at date 0, which serves only as an opportunity for consumers to sell capital goods and for bankers to choose their mix of deposits and equity.

Consumer preferences are given by the utility function

$$\sum_{t=1}^{\infty} \beta^t u(c_t),$$

where $0 < \beta < 1$ is the common discount factor, c_t denotes consumption at date t and $u(c)$ is the utility from consumption c. The function $u(\cdot)$ is assumed to satisfy the usual neoclassical properties.[17]

[16] We assume that φ is C^1 on $(0,\infty)$ and that $\varphi'(I) > 0$ and $\varphi''(I) < 0$, for $I > 0$, and $\lim_{I \searrow 0} \varphi'(I) = \infty$.

[17] That is, $u : \mathbf{R}_+ \to \mathbf{R}$ is C^2 on the interval $(0,\infty)$, $u'(c) > 0$ and $u''(c) \leq 0$, for all $c > 0$, and $u'(c) \to \infty$ as $c \searrow 0$.

2.2 Bankers

The face value of deposits issued by a representative bank at the end of period $t-1$ is denoted by $d_t k_t$, where d_t is the face value of deposits per unit of capital, and k_t is the capital stock.[18] A banker i can satisfy the depositors' demand for withdrawals in sub-period A of period t if and only if

$$\theta_{it} A k_t \geq d_t k_t,$$

where θ_{it} is the bank's productivity shock. This condition is equivalent to $\theta_{it} \geq z_t$, where $z_t \equiv d_t/A$ is the *breakeven level* of the productivity shock. In what follows, we treat z_t as the banker's decision variable.

Banks that are unable to meet the depositors' demand for withdrawals are considered to be in default, forced into bankruptcy, and liquidate their assets. The liquidation price of capital goods in sub-period B is denoted by q_t and the price of capital goods in sub-period C is denoted by p_t. Equilibrium requires that $q_t \leq p_t$. Otherwise, no one would buy capital goods in sub-period B. But q_t can be strictly less than p_t if there is insufficient market liquidity in sub-period B.

Note that banks that are unable to meet their depositors demands are illiquid, but not necessarily insolvent. Solvency would be the relevant criterion for avoiding bankruptcy if banks could borrow against their capital goods from liquid banks. A bank is solvent if

$$\theta_{it} A k_t + q_t k_t \geq d_t k_t,$$

or $\theta_{it} \geq (d_t - q_t)/A$, where q_t is the fire sale price of capital goods. Any solvent but illiquid bank would be able to meet its depositors' demands and avoid bankruptcy by borrowing on the interbank market. The illiquid banks' assets are valued at the fire sale price q_t: that is the price at which liquid banks can purchase capital goods and they will not be willing to lend for a smaller return. This is equivalent to having the fire sale in subperiod A. The quantitative results will differ if solvency rather than illiquidity is the criterion for failure, but the qualitative results will be the same.

In sub-period C, the solvent bankers are in possession of the entire stock of capital goods, having purchased the capital goods liquidated by the insolvent bankers in sub-period B. Their banks have no deposits, having paid out $d_t k_t$ in sub-period A. They choose new levels of deposits and equity to maximize the value of the bank. Because there is no aggregate uncertainty and consumers can diversify across banks, equity and deposits are priced as though consumers were risk neutral.

Suppose a bank has one unit of capital goods at the end of period t and issues deposits with face value d_{t+1}. The expected withdrawal from the bank in sub-period A of date $t+1$ is

$$R_A(z_{t+1}) = A \int^{z_{t+1}} \theta_{t+1} dF + A z_{t+1} \left(1 - F(z_{t+1})\right).$$

[18]We adopt the convention of scaling variables by the capital stock, since the scale factor is irrelevant for many purposes. Thus, d_t is the amount of deposits in a bank with one unit of capital goods under management. Since the bank's behavior is independent of scale, we might as well assume $k_t = 1$ when analyzing the bank's behavior.

The expected withdrawal $R_A(z_{t+1})$ is also the amount that a fully diversified depositor will withdraw for sure. In sub-period B, the capital goods of the insolvent banks are sold at the price q_t to the solvent banks. The value of the liquidated banks' assets is divided between the depositors and the equity holders and, since they cannot consume in sub-period B, they carry this amount forward to sub-period C and use it to purchase deposits and equity. The bank's expected returns in sub-period C of date $t+1$ are equal to

$$R_C(z_{t+1}) = q_{t+1}(1-\delta) F(z_{t+1}) + \frac{p_{t+1} A \int_{z_{t+1}} (\theta_{t+1} - z_{t+1})\, dF}{q_{t+1}} + p_{t+1}(1-\delta)(1 - F(z_{t+1})).$$

The first expression on the right hand side represents the value of liquidated assets, which is paid out to depositors and shareholders of the failed banks in sub-period B. This amount is carried forward and used to purchase deposits and equity in sub-period C. The second expression is the value of the capital goods purchased by solvent banks in sub-period B. Note that the capital goods are valued at p_{t+1}, the market price in sub-period C, because this represents their true economic value. The third and final expression is the value in sub-period C of the depreciated capital stock solvent banks brought forward from sub-period A.[19]

From the point of view of the bank, there is no need to distinguish between the amounts received by shareholders, on the one hand, and amounts received by depositors, on the other. All consumers hold the same portfolio of deposits and equity, receive the same payments from the bank, and value a dollar of income in a given subperiod the same, whether it is generated by equity or deposits.

Because consumption is only possible in sub-period A, the marginal utility of income in sub-period A may be greater than the marginal utility of income in sub-period C. For this reason, the bank's objective function, at the end of period t, is a weighted sum of $R_A(z_{t+1})$ and $R_C(z_{t+1})$,

$$\beta^{t+1} u'(c_{t+1}) R_A(z_{t+1}) + \psi_{t+1} R_C(z_{t+1}), \tag{1}$$

where $\beta^{t+1} u'(c_{t+1})$ is the marginal utility of income in sub-period A and ψ_{t+1} is the marginal utility of income in sub-period C. The bank chooses z_{t+1} at the end of date t to maximize (1).

[19] In equilibrium, the expression for $R_C(z_{t+1})$ will always be equal to the value of retained earnings plus the value of the depreciated capital goods,

$$A \int_{z_{t+1}} (\theta_{t+1} - z_{t+1})\, dF + p_{t+1}(1-\delta),$$

as we will see when we consider the market clearing conditions. The alternative definition we give here is the appropriate one for the individual banker as he considers possible deviations from the equilibrium value of z_{t+1}, taking prices as given.

Proposition 1 *If* $\beta^{t+1}u'(c_{t+1}) > 0$ *and* $\psi_{t+1} > 0$*, there is a unique value of* $z_{t+1} \in (0, \Theta)$ *that satisfies the first-order condition*

$$\beta^{t+1}u'(c_{t+1}) R'_A(z_{t+1}) + \psi_{t+1} R'_C(z_{t+1}) = 0.$$

Proof. See the appendix. ■

The uniqueness of the optimal capital structure follows from the strict concavity of the objective function. The strict concavity of the objective function in turn follows from the increasing hazard rate property of the cumulative distribution function F.

—— Insert Figure 2 here ——

Figure 2 illustrate the flows of goods and financial assets among the banks, producers and consumers.

2.3 Capital producers

The capital goods producers maximize profits each period, so that

$$\pi_t = \sup_{I_t \geq 0} \{p_t \varphi(I_t) - I_t\}.$$

Under the maintained assumptions, for any positive price p_t, the optimal investment I_t is strictly positive and is uniquely determined by the first-order condition

$$p_t \varphi'(I_t) = 1. \tag{2}$$

2.4 Consumers

We have left the consumer's problem until last because it is mainly used to determine the prices at which the representative consumer is willing to hold the equilibrium portfolio of deposits and equity.

Deposits play a special role in the economy because of our assumption that only deposits can be used for consumption. It is sometimes argued that this property of bank deposits depends on their "safety" (cf. Stein, 2012), but our deposits are apparently risky. There is no aggregate uncertainty, however, and so the individual depositor can avoid any risk by diversifying his holdings across a large number of banks subject to independent shocks. In fact, diversification has the same effect as deposit insurance. Suppose that the Deposit Insurance Corporation guarantees withdrawals up to a fraction λ of the face value of deposits and levies a premium equal to $\max\{\min\{\theta A/d, 1\} - \lambda, 0\}$ on each dollar of deposits. Then each consumer who invests in a single bank will be guaranteed a withdrawal of exactly λdk. Banks that pay a premium lower than $1-\lambda$ are considered to be bankrupt and are liquidated.

We can assume without loss of generality that each consumer holds the minimum amount of deposits that he needs for his planned consumption next period and holds the rest of his wealth as equity. As we shall see, the return on (fully diversified) deposits is never greater than the return on equity, so it is optimal to maximize the amount of equity in the consumer's portfolio, subject to the constraint that he has enough deposits to fund his desired consumption.

To describe the representative consumer's decision problem, we need to introduce the following notation. The consumer's demand for deposits, expressed as a multiple of the capital stock, is denoted by d_t and the demand for equity, expressed as a multiple of the capital stock, is denoted by e_t. Thus, the face value of deposits at date t is $d_t k_t$ and the number of shares outstanding is $e_t k_t$. Since the capital stock is taken as given by the consumer, his decision variables are d_t and e_t.

The consumer's budget constraints in period t reflect the division of activities among the sub-periods. In sub-period A, the consumer withdraws deposits and consumes them. In sub-period B, the consumer receives the proceeds from the sale of capital goods by banks in liquidation. These proceeds are held until the last period when they can be invested in shares and deposits. There are thus two budget constraints, one corresponding to the activity in sub-period A and one corresponding to the activity in sub-period C.

In sub-period A, the consumer does not receive the full face value of his deposits because some banks default. A consumer holding one unit of deposits, fully diversified across all banks at the end of date t, receives λ_t in sub-period A at date t. His budget constraint in sub-period A will be

$$c_t \leq \lambda_t d_t k_t.$$

As noted above, we can assume without loss of generality that this constraint holds as an equation in equilibrium and use it to define consumption.

The additional payment that depositors receive from liquidated banks in sub-period B is denoted by μ_t. This amount cannot be consumed immediately but it can be used to purchase deposits and equity in sub-period C. The total return at date t to a unit of equity purchased at date $t-1$ is denoted by R_t. Finally, π_t denotes the profits from producing new capital goods, which are paid immediately to consumers in sub-period C. At date 0, the consumer's wealth is k_0 if we normalize the price of capital goods to 1. At any date $t > 0$, in sub-period C, the consumer's wealth is

$$w_t = \mu_t d_t k_t + e_t k_t R_t + \pi_t.$$

The price of one unit of deposits at date t is denoted by r_t and the price of one of equity at date t is denoted by v_t. At any date t, the consumer's expenditure on deposits is $r_t d_{t+1} k_{t+1}$ and the expenditure on equity is $v_t e_{t+1} k_{t+1}$, so the budget constraint in sub-period C is

$$r_0 d_1 k_1 + v_0 e_1 k_1 \leq k_0,$$

at date 0, and

$$r_t d_{t+1} k_{t+1} + v_t e_{t+1} k_{t+1} \leq \mu_t d_t k_t + e_t k_t R_t + \pi_t,$$

at any date $t > 0$.

The consumer's problem is to choose the non-negative sequence $\{(c_t, d_t, e_t)\}_{t=1}^{\infty}$ to solve the decision problem

$$\begin{array}{rl} \max & \sum_{t=1}^{\infty} \beta^t u(c_t) \\ \text{s.t.} & c_t \leq \lambda_t d_t k_t, \ \forall t \geq 1 \\ & r_0 d_1 k_1 + v_0 e_1 k_1 \leq k_0 \\ & r_t d_{t+1} k_{t+1} + v_t e_{t+1} k_{t+1} \leq \mu_t d_t k_t + e_t k_t R_t + \pi_t, \ \forall t \geq 1. \end{array} \tag{3}$$

Assuming an interior solution, the first-order conditions are

$$\beta^{t+1} u'(c_{t+1}) \lambda_{t+1} = \psi_t r_t - \psi_{t+1} \mu_{t+1}, \tag{4}$$

and

$$\psi_t v_t = \psi_{t+1} R_{t+1}, \tag{5}$$

for every t, where ψ_t is the Lagrange multiplier on the budget constraint in sub-period C at date t.

An increase of one unit in deposits increases utility by $\beta^{t+1} u'(c_{t+1}) \lambda_{t+1} k_{t+1}$ at date $t+1$, but it also increases the budget constraint by $r_t k_{t+1}$ at date t and reduces it by $\mu_{t+1} k_{t+1}$ at date $t+1$. If ψ_t and ψ_{t+1} are the Lagrange multipliers of the budget constraints in sub-period C at dates t and $t+1$, the optimal choice of deposits requires

$$\beta^{t+1} u'(c_{t+1}) \lambda_{t+1} k_{t+1} = \psi_t r_t k_{t+1} - \psi_{t+1} \mu_{t+1} k_{t+1}.$$

Dividing both sides by k_{t+1} yields (4). The first-order condition (5) for the choice of equity follows similarly. The left hand side $\psi_t v_t$ is the price of one unit of equity times the marginal utility of money in sub-period C at date t, and the right hand side is the return to one unit of equity times the marginal utility of money in sub-period C at date $t+1$.

Note that the total returns to equity, $e_t k_t R_t$, consist of the earnings of liquid banks, $Ak_t \int_{z_t} (\theta_t - z_t)\, dF = I_t$, minus the payments to depositors $\mu_t d_t k_t$, plus the value of the capital goods purchased in sub-period B and those retained by the liquid banks, $p_t (1 - \delta) k_t$. Then the consumer's wealth is

$$\begin{aligned} \mu_t d_t k_t + e_t k_t R_t + \pi_t &= \mu_t d_t k_t + I_t - \mu_t d_t k_t + p_t \varphi(I_t) - I_t + p_t (1 - \delta) k_t \\ &= p_t k_{t+1}. \end{aligned}$$

So the market value of a bank with k_{t+1} units of capital goods, $r_t d_{t+1} k_{t+1} + v_t e_{t+1} k_{t+1}$, equals the value of the assets it holds, $p_t k_{t+1}$.

Because the consumer's optimization problem is convex, the solution can be characterized as a sequence $\{(c_t, d_t, e_t)\}_{t=0}^{\infty}$ that satisfies the budget constraints in (3) and first-order conditions (4) and (5), for every t.

2.5 Market clearing

Sub-period A: Recall that bankers can meet their depositors' demands in sub-period A of date t if and only if the productivity shock $\theta_{it} \geq z_t$. If $\theta_{it} < z_t$, the bank will pay out its entire revenue $\theta_{it}Ak_t$; if $\theta_{it} \geq z_t$, the bank will pay out $d_t k_t = z_t A k_t$. The total amount withdrawn from bank i at date t is

$$\min\{d_t k_t, \theta_{it} A k_t\} = \min\{z_t, \theta_{it}\} A k_t.$$

Risk averse consumers will diversify their deposits across the continuum of banks to avoid the idiosyncratic risk of each bank. A fully diversified consumer will therefore be able to consume

$$\begin{aligned} c_t &= \int \min\{z_t, \theta_{it}\} A k_t di \\ &= \int \min\{z_t, \theta_t\} A k_t dF \\ &= A k_t \left\{ \int^{z_t} \theta dF + z_t (1 - F(z_t)) \right\}. \end{aligned}$$

in sub-period A at date t, where we assume without loss of generality that the cash-in-advance constraint holds as an equation.

Sub-period B: The liquidation price of capital goods in sub-period B is denoted by q_t. The amount of capital goods to be liquidated is $(1-\delta)k_t F(z_t)$. The only source of funds to purchase capital goods is the solvent banks' retained earnings

$$\int_{z_t} (\theta_t A - d_t) k_t dF = A k_t \int_{z_t} (\theta_t - z_t) dF.$$

In equilibrium, we must have

$$q_t \leq p_t.$$

Otherwise, no bank would be willing to buy liquidated capital goods. If $q_t < p_t$, liquid banks will use all their spare cash to purchase liquidated capital. There are two ways that the market can clear in sub-period B. Either $q_t = p_t$ and

$$q_t(1-\delta)k_t F(z_t) \leq A k_t \int_{z_t} (\theta_t - z_t) dF$$

or $q_t < p_t$ and

$$q_t(1-\delta)k_t F(z_t) = A k_t \int_{z_t} (\theta_t - z_t) dF.$$

Since the capital stock k_t appears on both sides of these conditions, it can be eliminated and q_t can be expressed as a function of z_t:

$$q_t = \min\left\{ \frac{A \int_{z_t} (\theta_t - z_t) dF}{(1-\delta)F(z_t)}, p_t \right\}.$$

Sub-period C: In sub-period C, the total amount of the consumption good available is $Ak_t \int_{z_t} (\theta_t - z_t)\, dF$.[20] Since the consumption good cannot be stored, it must be used as an input for the production of capital goods. Thus, market clearing requires

$$I_t = Ak_t \int_{z_t} (\theta_t - z_t)\, dF.$$

The capital stock is determined by the initial condition $k_1 = k_0$ and the law of motion

$$k_{t+1} = (1 - \delta)\, k_t + \varphi (I_t)$$

for any $t \geq 1$.

2.6 Equilibrium

An *attainable allocation* for the economy is a sequence $\{(c_t, d_t, e_t, I_t, k_t, z_t)\}_{t=1}^{\infty}$ satisfying the following conditions: *first*, the equilibrium values are non-negative,

$$(c_t, d_t, e_t, I_t, k_t, z_t) \geq \mathbf{0}, \text{ for any } t = 1, \dots\ , \tag{6}$$

second, consumption is equal to the total amount withdrawn from banks,

$$c_t = Ak_t \left\{ \int_0^{z_t} \theta dF + z_t (1 - F(z_t)) \right\}, \text{ for any } t = 1, \dots\ , \tag{7}$$

third, investment in new capital goods is equal to the earnings retained by solvent banks,

$$I_t = Ak_t \int_{z_t} (\theta_t - z_t)\, dF, \text{ for any } t = 1, \dots\ , \tag{8}$$

and, *fourth*, the new capital stock is equal to the depreciated capital stock plus the output of new capital goods,

$$k_{t+1} = (1 - \delta)\, k_t + \varphi (I_t)\, , \text{ for any } t = 1, \dots\ , \tag{9}$$

where $k_1 = k_0$.

For any attainable allocation $\{(c_t, d_t, e_t, I_t, k_t, z_t)\}_{t=1}^{\infty}$, an *admissible price system* is a sequence $\{(p_t, q_t, \psi_t)\}_{t=0}^{\infty}$ such that prices are non-negative

$$(p_t, q_t, \psi_t) \geq \mathbf{0}, \text{ for any } t = 0, 1, \dots\ , \tag{10}$$

and the price of liquidated capital goods is the minimum of the cash-in-the-market price in sub-period B and the market clearing price in sub-period C,

$$q_t = \min \left\{ \frac{Ak_t \int_{z_t} (\theta_t - z_t)\, dF}{(1 - \delta)\, F(z_t)}, p_t \right\}, \text{ for any } t = 1, \dots. \tag{11}$$

[20] The same stock of goods is first retained by solvent banks as earnings in subperiod A, then used to purchase liquidated capital goods from failed banks, who in turn pay their depositors and shareholders, in subperiod B, and finally used to produce capital goods in subperiod C.

An *equilibrium* consists of an attainable allocation $\{(c_t, d_t, e_t, I_t, k_t, z_t)\}_{t=1}^{\infty}$ and an admissible price system $\{(p_t, q_t, \psi_t)\}_{t=0}^{\infty}$, such that $\{(d_t, e_t)\}_{t=1}^{\infty}$ solves the consumer's problem, $\{(z_{t+1})\}_{t=1}^{\infty}$ solves the banker's problem and $\{I_t\}_{t=1}^{\infty}$ solves the capital producer's problem, where the auxiliary variables λ_{t+1}, μ_{t+1} and R_{t+1} are defined by

$$\lambda_{t+1} = A \int^{z_{t+1}} \theta dF + A z_{t+1} \left(1 - F\left(z_{t+1}\right)\right),$$

$$\mu_{t+1} = A k_t \int_0^{z_t} \min\left\{z_t - \theta_t, q_t \left(1 - \delta\right)\right\} dF,$$

and

$$R_{t+1} = \frac{I_{t+1} - \mu_t d_{t+1} k_{t+1} + p_{t+1} \left(1 - \delta\right) k_{t+1}}{e_{t+1} k_{t+1}}.$$

3 Pareto efficiency

Because there is a representative agent, an attainable allocation is *Pareto efficient* if it maximizes the consumer's utility. We can characterize a Pareto efficient allocation as the solution to a planner's problem. A *feasible solution* for the planner's problem is a sequence $\{(c_t, I_t, k_t)\}_{t=1}^{\infty} \in \mathbf{R}_+^3$ such that

$$c_t + I_t = A k_t, \ \forall t, \tag{12}$$

$$k_{t+1} = \left(1 - \delta\right) k_t + \varphi\left(I_t\right), \ \forall t, \tag{13}$$

$$k_1 = k_0. \tag{14}$$

An *optimal solution* of the planner's problem is a feasible solution that maximizes the utility of the representative consumer

$$\sum_{t=1}^{\infty} \beta^t u\left(c_t\right).$$

The properties of the optimal solution to the planner's problem are summarized in the following result.

Theorem 2 *Under the maintained assumptions, a feasible solution $\{(c_t, I_t, k_t)\}_{t=1}^{\infty}$ is optimal if and only if*

$$\begin{aligned} \left(c_t, I_t, k_t\right) &\gg \mathbf{0}, \ \forall t, \\ \left(\alpha_t, \gamma_t\right) &\gg \mathbf{0}, \ \forall t, \end{aligned}$$

and there is a sequence of multipliers $\{(\alpha_t, \gamma_t)\}_{t=1}^{\infty}$ such that

$$-\alpha_t^* + \beta^t u'\left(c_t\right) = 0, \ \forall t, \tag{15}$$

$$-\alpha_t^* + \gamma_t^* \varphi'\left(I_t\right) = 0, \ \forall t, \tag{16}$$

$$\alpha_t^* A + \gamma_t^* \left(1 - \delta\right) - \gamma_{t-1}^* = 0, \ \forall t, \tag{17}$$

The optimal solution $\{(c_t, I_t, k_t)\}_{t=1}^{\infty}$ converges monotonically to a steady state (c^*, I^*, k^*).

Proof. See the appendix. ■

The first-order conditions of the planner's problem are quite intuitive. The parameters (α_t, γ_t) are the Lagrange multipliers on the constraints (12) and (13), respectively. Equation (15) says that the marginal utility of consumption in period t is equal to α_t. Equation (16) says that the marginal utility of consumption is equal to the marginal utility of capital, γ_t, times the amount of capital produced by one unit of consumption. Equation (17) says that the marginal utility of capital at date $t-1$, γ_{t-1}, is equal to the marginal utility of consumption at t, α_t, times the consumption produced by one unit of capital, plus the marginal utility of capital at t, γ_t, times the fraction of capital remaining after depreciation. Alternatively, we can think of $p_t = \gamma_t^* / \alpha_t^*$ as the shadow price of one unit of capital, measured in units of the consumption good. Then the first-order condition (16) can be rewritten as

$$p_t \varphi' (I_t) = 1,$$

which is the first-order condition for maximizing profit $p_t \varphi (I_t) - I_t$. And the first-order condition (17) can be rewritten as

$$p_{t-1} = A + p_t (1 - \delta),$$

which says the value of one unit of capital at date $t-1$ is equal to the output at date t plus the value of the (depreciated) capital good.

3.1 Optimal policy

An equilibrium allocation need not be Pareto efficient. An equilibrium is constrained in a number of ways that the planner's problem is not. *First*, it is constrained by a transaction technology that requires consumers to use deposits to obtain consumption. *Second*, banks are constrained to use deposits and equity to fund their balance sheets and may be forced into bankruptcy and liquidation if they cannot meet their depositors demands. The relevant question is whether the planner could do better than the market (laisser-faire equilibrium) when similarly constrained by the same market technology. A concept of *constrained efficiency*, in which the planner is subject to the same technology as the market, would seem to be the appropriate benchmark. Suppose, for example, that the planner can only control the breakeven level z_t at each date t, that banks, producers and consumers behave optimally, and prices adjust to clear markets. Would the planner do better than the laisser faire equilibrium under these conditions? Surprisingly, it turns outs that, even under these constraints, the planner can achieve the first best.

This result is partly explained by the observation that the transactions constraint, which requires consumers to use deposits to obtain consumption, does not actually restrict the planner's choices. To see this, consider the following constrained version of the planner's problem. A feasible allocation for the *constrained planner's problem* is a sequence

$\{(c_t, I_t, k_t, z_t)\}_{t=1}^{\infty} \in \mathbf{R}_+^3$ such that

$$c_t = Ak_t \int^{z_t} \theta_t dF + z_t A k_t (1 - F(z_t)) \tag{18}$$

$$I_t = Ak_t \int_{z_t} (\theta_t - z_t) \, dF \tag{19}$$

$$k_{t+1} = (1 - \delta) k_t + \varphi(I_t), \ \forall t, \tag{20}$$

$$k_1 = k_0. \tag{21}$$

An optimal solution of the constrained planner's problem is a feasible solution (for the constrained problem) that maximizes the utility of the representative consumer. Now suppose that $\{(c_t^*, I_t^*, k_t^*)\}_{t=1}^{\infty}$ is a solution to the (unconstrained) planner's problem and, for each date t, define z_t^* so that

$$c_t^* = Ak_t^* \int^{z_t^*} \theta_t dF + z_t^* A k_t^* (1 - F(z_t^*)).$$

It is clear that z_t^* is well defined, because the right hand side of this equation varies continuously from 0 to Ak_t^* as z_t varies from 0 to Θ and the feasibility constraints of the planner's problem ensure that $0 \leq c_t^* \leq Ak_t^*$. The feasibility conditions of the planner's problem also imply that

$$I_t^* = Ak_t^* \int_{z_t^*} (z_t^* - \theta_t) \, dF,$$

so $\{(c_t^*, I_t^*, k_t^*, z_t^*)\}_{t=1}^{\infty}$ is a feasible solution of the constrained planner's problem and achieves the first best welfare level. It is obvious that no feasible solution of the constrained planner's problem can achieve a higher level of welfare, so the constrained and unconstrained problems are equivalent.

This is not the end of the matter, however. We still have to show that it is sufficient for the planner to control the banks' capital structure, that is, we have to show that it is possible for the solution to the planner's constrained problem to be decentralized.

An *equilibrium relative to the breakeven levels* $\{\bar{z}_t\}_{t=1}^{\infty}$ is an attainable allocation $\{(c_t, d_t, e_t, I_t, k_t, z_t)\}_{t=1}^{\infty}$ and an admissible price system $\{(p_t, q_t, r_t, v_t, \psi_t)\}_{t=0}^{\infty}$ such that $\{(d_t, e_t)\}_{t=1}^{\infty}$ solves the consumer's problem, $\{I_t\}_{t=1}^{\infty}$ solves the capital producer's problem, and

$$z_t = \bar{z}_t, \ \forall t,$$

where the auxiliary variables λ_{t+1}, μ_{t+1} and R_{t+1} are defined in the usual way (see Section 2.6). In a relative equilibrium, the regulator chooses a sequence of breakeven levels $\{\bar{z}_t\}_{t=1}^{\infty}$, which determines the capital structure at each date. Then prices adjust to clear markets and bankers, producers and consumers maximize their objectives, taking prices and the regulator's policy as given. The next result shows that, by setting the right breakeven levels, a regulator can implement the first best as an equilibrium.

Theorem 3 *If $\{(c_t^*, k_t^*, I_t^*, \alpha_t^*, \gamma_t^*)\}_{t=0}^{\infty}$ is the solution to the planner's problem defined by (12) through (14), there exists a sequence of breakeven levels $\{z_t^*\}_{t=0}^{\infty}$ and an equilibrium $\{(c_t, d_t, e_t, z_t, k_t, I_t, p_t, q_t, r_t, v_t)\}_{t=0}^{\infty}$ relative to $\{z_t^*\}_{t=0}^{\infty}$ such that $(c_t, k_t, I_t) = (c_t^*, k_t^*, I_t^*)$ for every t.*

Proof. See the appendix. ■

Because the model is relatively simple, the planner's ability to control the capital structure essentially determines the entire allocation of resources in the economy. First, the banks' capital structure at each date is determined by the breakeven level. Second, it is clear from the feasibility conditions (18) through (21) that, given the capital stock k_t inherited from the past, the choice of z_t determines the level of consumption c_t, investment I_t and the new capital stock k_{t+1}. Hence, starting with the initial capital stock k_0, the choice of capital structures $\{z_t\}$ recursively determines the entire allocation $\{(c_t, d_t, k_t, I_t)\}$. Then, in order to show that this allocation is an equilibrium relative to the capital structures $\{z_t\}$, all we need to do is find prices at which consumers are willing to hold the appropriate deposits and equity and the producers are willing to make the right investments and produce the right amount of capital goods.

4 Efficient equilibria

An equilibrium is not necessarily Pareto efficient and, hence, not constrained efficient, but there are conditions under which it may be both. We begin this section by identifying a sufficient condition for an equilibrium to be Pareto and constrained efficient and then show that this condition is necessary as well.

4.1 A sufficient condition for efficiency of banking equilibrium

The following result describes three equivalent statements of a sufficient condition for equilibrium to be efficient.

Proposition 4 *Let $\{(c_t^*, d_t^*, e_t^*, I_t^*, k_t^*, z_t^*)\}_{t=1}^{\infty}$ and $\{(p_t^*, q_t^*, \psi_t^*)\}_{t=1}^{\infty}$ be a banking equilibrium. The following conditions are equivalent:*

$$\begin{array}{lll} (i) & \beta^{t+1} u'\left(c_{t+1}^*\right) = \psi_{t+1}^*, & \forall t, \\ (ii) & p_t^* = q_t^*, & \forall t, \\ (iii) & (1-\delta)\, p_t^* k_t^* F\left(z_t^*\right) \le I_t^*, & \forall t. \end{array}$$

Proof. See the appendix. ■

Condition (i) says that marginal utility of income is the same in sub-periods A and C. Condition (ii) says that the price of capital goods is the same in sub-periods B and C. Condition (iii) says that the amount of cash in the market in sub-period B is greater than or equal to the fundamental value of the capital goods that are liquidated in sub-period B. The equivalence of Conditions (ii) and (iii) is immediate from the market clearing condition in sub-period B. The equivalence of Condition (i) with the other two is more subtle, but intuitive. If the marginal utility of income in sub-periods A and C are equal, then the "cash-in-advance" constraint in sub-period A has a zero Lagrange multiplier. If default were costly ($p_t^* > q_t^*$), it would be optimal for the banks to lower z_t^* to take advantage of the capital gains from buying capital goods in sub-period B. Conversely, if default were not costly ($q_t^* = p_t^*$), and the marginal utility of income were higher in sub-period A than in sub-period B, banks would increase z_t^* in order to increase consumption in sub-period A.

The next result shows that, in the absence of costly default or, equivalently, a binding "cash-in-advance" constraint, equilibrium is efficient.

Theorem 5 *Let* $\{(c_t^*, d_t^*, e_t^*, I_t^*, k_t^*, z_t^*, p_t^*, q_t^*, \psi_t^*)\}_{t=1}^{\infty}$ *be a banking equilibrium and suppose that one of the conditions in Proposition 4 is satisfied. Then the allocation* $\{(c_t^*, d_t^*, e_t^*, I_t^*, k_t^*, z_t^*)\}_{t=1}^{\infty}$ *is Pareto efficient.*

Proof. If $\beta^{t+1}u'\left(c_{t+1}^*\right) = \psi_{t+1}^*$, the first-order conditions of the consumer's problem imply that, for all t,

$$u'\left(c_t^*\right) p_t^* = \beta u'\left(c_{t+1}^*\right) \left\{A + p_{t+1}^*\left(1-\delta\right)\right\}.$$

But setting

$$\alpha_t^* = \beta^t u'\left(c_t^*\right)$$

and

$$\gamma_t^* = \alpha_t^* p_t^*$$

for all t, gives us the first-order conditions for the planner's problem:

$$\begin{aligned}
\beta^t u'\left(c_t^*\right) &= \alpha_t^*, \ \forall t, \\
\gamma_t^* \varphi'\left(I_t^*\right) &= \alpha_t^*, \ \forall t, \\
\gamma_t^* &= \alpha_{t+1}^* A + \gamma_{t+1}^*\left(1-\delta\right), \ \forall t.
\end{aligned}$$

This proves the desired result. ■

In short, if the "cash-in-advance" constraint is not binding and default is not costly, the allocation of output between consumption and investment is not distorted.

4.2 A necessary condition for efficiency of banking equilibrium

The proof that the condition is necessary as well as sufficient is more difficult, but it follows the same logic. We assume, contrary to what we want to prove, that the equilibrium is efficient even though the sufficient condition is violated at some date. We then show that the condition must be violated at all dates and, hence, must be violated in the steady-state to which the efficient allocation converges. Then it is easy to show that a necessary condition for efficiency of the steady state is not satisfied.

Theorem 6 *Let* $\{(c_t^*, d_t^*, e_t^*, I_t^*, k_t^*, z_t^*, p_t^*, q_t^*, \psi_t^*)\}_{t=1}^{\infty}$ *be a banking equilibrium and suppose that the allocation* $\{(c_t^*, d_t^*, e_t^*, I_t^*, k_t^*, z_t^*)\}_{t=1}^{\infty}$ *is Pareto efficient. Then the conditions in Proposition 4 must be satisfied for every* t.

Proof. See the appendix. ■

This theorem tells us that costly fire sales ($q_t < p_t$) are a necessary and sufficient condition for constrained inefficiency of equilibrium. From the banks' point of view, it is easy to see why fire sales are costly and ought to be avoided. If we write out the banks' objective function in full we get

$$\beta^{t+1} u'(c_{t+1}) \left\{ A \int^{z_{t+1}} \theta_{t+1} dF + A z_{t+1} (1 - F(z_{t+1})) \right\}$$
$$+ \psi_{t+1} \left\{ (q_{t+1} - p_{t+1})(1-\delta) F(z_{t+1}) + \frac{p_{t+1} A \int_{z_{t+1}} (\theta_{t+1} - z_{t+1}) dF}{q_{t+1}} + p_{t+1}(1-\delta) \right\}.$$

An increase in z_{t+1} clearly increases the expected value of deposits in sub-period A at date $t+1$ (the first term in the expression above). An increase in z_{t+1} will have two effects on the second term. First, since $q_{t+1} \leq p_{t+1}$, an increase in z_{t+1} must decrease or leave unchanged

$$(q_{t+1} - p_{t+1})(1-\delta) F(z_{t+1}).$$

Second, an increase in z_{t+1} must decrease the expected value of retained earnings

$$A \int_{z_{t+1}} (\theta_{t+1} - z_{t+1}) dF,$$

for the same reason it increases the expected value of deposits. Thus, an increase in z_{t+1} decreases the second expression. The analysis of the costs and benefits of increasing z_{t+1} includes two 'costs' that would not be evident to a central planner. These are the *increasing* likelihood of losing money in a fire sale and the *decreasing* likelihood of making a profit by remaining solvent and buying up assets in the fire sale.

While the fire sale of assets represents a genuine *private* cost that a value maximizing bank should try to avoid, it does not represent a *social* cost. The shareholders and depositors of a defaulting bank lose value when assets are sold at less than their economic value. This loss is a capital gain for the shareholders of the banks that purchase the assets. This becomes clear if we look at the bank's objective function from the point of view of the planner. The planner recognizes that an increase in z_{t+1} (for all banks) changes the price q_{t+1} so that capital gains and losses from the fire sales offset each other. If we substitute from the market-clearing condition in sub-period B, the objective function becomes

$$\begin{aligned}
& \beta^{t+1}u'(c_{t+1})\left\{A\int^{z_{t+1}}\theta_{t+1}dF+Az_{t+1}(1-F(z_{t+1}))\right\}+ \\
& \psi_{t+1}\left\{A\int_{z_{t+1}}(\theta_{t+1}-z_{t+1})\,dF+p_{t+1}(1-\delta)\,F(z_{t+1})+p_{t+1}(1-\delta)(1-F(z_{t+1}))\right\} \\
= \; & \beta^{t+1}u'(c_{t+1})\left\{A\int^{z_{t+1}}\theta_{t+1}dF+Az_{t+1}(1-F(z_{t+1}))\right\} \\
& \qquad +\psi_{t+1}\left\{A\int_{z_{t+1}}(\theta_{t+1}-z_{t+1})\,dF+p_{t+1}(1-\delta)\right\} \\
= \; & \beta^{t+1}u'(c_{t+1})\,A\int\theta_{t+1}dF+\left(\psi_{t+1}-\beta^{t+1}u'(c_{t+1})\right)A\int_{z_{t+1}}(\theta_{t+1}-z_{t+1})\,dF \\
& \qquad +\psi_{t+1}p_{t+1}(1-\delta)\,.
\end{aligned}$$

The price of liquidated capital goods, q_{t+1}, has disappeared, the breakeven level z_{t+1} only appears in the second expression

$$\left(\psi_{t+1}-\beta^{t+1}u'(c_{t+1})\right)A\int_{z_{t+1}}(\theta_{t+1}-z_{t+1})\,dF$$

and only has an impact on the objective function if $\psi_{t+1}-\beta^{t+1}u'(c_{t+1})<0$, in which case an increase in z_{t+1} increases the objective function. So the planner will continue to raise z_t until $\psi_{t+1}=\beta^{t+1}u'(c_{t+1})$ and efficiency is reached.

Although the 'costs' of fire sales are not evident to the planner, they are crucial to the determination of the bank's capital structure in equilibrium. Banks try to avoid these costs by using less than the efficient amount of deposit funding. This distortion causes the interest on deposits to fall until deposits become a sufficiently cheap source of funding to compensate for the risk of 'costly' default. The restriction of the supply of deposits also raises the cost of consumption, reducing consumption below the efficient level and increasing the share of output devoted to investment in new capital goods.

5 Overaccumulation

The ultimate effect of the cheap funding of banks by deposits is an inefficient overaccumulation of capital. Compared to the efficient path, an inefficient equilibrium path will have a higher capital stock and higher consumption and investment. To demonstrate this result, we focus on steady state outcomes. Consider first the steady-state of the efficient path $\{(c_t^*, I_t^*, k_t^*)\}_{t=1}^{\infty}$. We have shown that the optimal solution to the planner's problem converges to a steady state:

$$\lim_{t\to\infty}(c_t^*, I_t^*, k_t^*)=(c^*, I^*, k^*)$$

and

$$\lim_{t\to\infty} \beta^{-t} \left(\alpha_t^*, \gamma_t^*\right) = \left(\alpha^*, \gamma^*\right).$$

This steady state will satisfy the feasibility conditions

$$\begin{gathered} c^* + I^* = Ak^*, \\ k^* = (1-\delta)\,k^* + \varphi\left(I^*\right), \end{gathered}$$

and the first-order conditions

$$\begin{aligned} -\alpha^* + u'\left(c^*\right) &= 0, \\ -\alpha^* + \gamma^*\varphi'\left(I^*\right) &= 0, \\ \beta\alpha^* A + \beta\gamma^*\left(1-\delta\right) - \gamma^* &= 0. \end{aligned}$$

In fact, the first-order and the feasibility conditions define the steady state:

$$p^* = \frac{\beta A}{1-\beta\left(1-\delta\right)} \tag{22}$$

$$p^*\varphi'\left(I^*\right) = 1 \tag{23}$$

$$c^* + I^* = Ak^* \tag{24}$$

$$\delta k^* = \varphi\left(I^*\right), \tag{25}$$

where we write $p^* \equiv \alpha^*/\gamma^*$ and eliminate $u'\left(c^*\right)$.

Let $\{(c_t, d_t, e_t, I_t, k_t, z_t, p_t, q_t, \psi_t)\}_{t=1}^{\infty}$ be an inefficient equilibrium and suppose it also converges to a steady state

$$\lim_{t\to\infty} \left(c_t, d_t, e_t, I_t, k_t, z_t, p_t, q_t, \psi_t\right) = \left(c^0, d^0, e^0, I^0, k^0, z^0, p^0, q^0, \psi^0\right).$$

The array $\left(c^0, d^0, e^0, I^0, k^0, z^0, p^0, q^0, \psi^0\right)$ satisfies the first-order condition for the banker's problem,[21]

$$A + \psi^0 \left[\left(q^0 - p^0\right)\left(1-\delta\right)\frac{F'\left(z^0\right)}{1-F\left(z^0\right)} - \frac{p^0 A}{q^0}\right] = 0,$$

the first-order conditions on the consumer's problem,[22]

$$\psi^0 r^0 d^0 k^0 = \beta\left(c^0 + \psi^0\mu^0 d^0 k^0\right)$$

and

$$\psi^0 q^0 k^0 = \beta\psi^0\left(I - \mu^0 d^0 k^0 + p^0\left(1-\delta\right)k^0\right),$$

[21] Because $\beta^{t+1}u'\left(c_{t+1}\right)$ and ψ_{t+1} converge to zero as $t\to\infty$, we divide the first order condition by $\beta^{t+1}u'\left(c_{t+1}\right)$ and let ψ^0 denote the limit of the ratio $\psi_{t+1}/\beta^{t+1}u'\left(c_{t+1}\right)$.

[22] As in the case of the banker's first-order condition, we divide the first-order conditions by $\beta^{t+1}u'\left(c_{t+1}\right)$ and let ψ^0 denote the limit of the ratio $\psi_{t+1}/\beta^{t+1}u'\left(c_{t+1}\right)$.

which can be summarized by the pricing kernel[23]

$$\psi^0 p^0 k^0 = \beta \left(c^0 + \psi^0 \left(I^0 + p^0 (1-\delta) k^0\right)\right),$$

the first-order condition for the producer's problem

$$p^0 \varphi' \left(I^0\right) = 1,$$

and the market-clearing conditions

$$\begin{aligned} q^0 &= \frac{I^0}{(1-\delta) F\left(z^0\right) k^0}, \\ c^0 + I^0 &= Ak^0, \\ \delta k &= \varphi\left(I^0\right). \end{aligned}$$

We can select a subsystem from these equations that looks very similar to the system of equations defining the steady state of the planner's optimal solution:

$$\psi^0 p^0 k^0 = \frac{\beta\left(c^0 + \psi^0 I^0\right)}{1-\beta(1-\delta)} \tag{26}$$

$$p^0 \varphi'\left(I^0\right) = 1, \tag{27}$$

$$c^0 + I^0 = Ak^0, \tag{28}$$

$$\delta k^0 = \varphi\left(I^0\right). \tag{29}$$

The first equation is simply a rearrangement of the pricing kernel and the last three are the producer's first-order condition and two of the market-clearing conditions. Comparing the system of equations (22-25) with the system of equations (26-29), the second, third and fourth equations are the same. In fact, equation (26) is the same as equation (22), if we put $\psi = 1$, because

$$\begin{aligned} pk &= \frac{\beta(c+I)}{1-\beta(1-\delta)} \\ &= \frac{\beta Ak}{1-\beta(1-\delta)} \end{aligned}$$

which is equivalent to (22) after dividing both sides by k. The next theorem tells us that the system of four equations determines the four unknowns (c, I, k, p) as functions of the parameter ψ. Then allowing the parameter ψ to vary from ψ^0 to 1, we can trace out the values of (c, I, k, p) as they vary from (c^0, I^0, k^0, p^0) to (c^*, I^*, k^*, p^*).

[23] This equation is obtained by summing the first-order conditions for the consumer's problem.

Theorem 7 *For any $\psi^0 \leq \psi \leq 1$, the system of equations*

$$\begin{aligned}
\psi pk &= \frac{\beta\left(c+\psi I\right)}{1-\beta\left(1-\delta\right)} \\
p\varphi'\left(I\right) &= 1, \\
c+I &= Ak, \\
\delta k &= \varphi\left(I\right).
\end{aligned}$$

has a unique solution $(c, I, k, p) = \Phi(\psi)$, where $\Phi : [\psi^0, 1] \to \mathbf{R}^4_+$ is a C^1 function,

$$\Phi\left(\psi^0\right) = \left(c^0, I^0, k^0, p^0\right)$$

and

$$\Phi\left(\psi^*\right) = \left(c^*, I^*, k^*, p^*\right).$$

Proof. See the appendix. ■

The next step is to use the local comparative static properties to compare the steady state values (c^0, I^0, k^0, p^0) and (c^*, I^*, k^*, p^*).

Proposition 8 *For any value of ψ in the interval $\left[\psi^0, 1\right]$, the function $(I, k, p) = (\Phi_I(\psi), \Phi_k(\psi), \Phi_p(\psi))$ is strictly decreasing.*

Proof. See the appendix. ■

The proposition ensures us that

$$\left(c^0, I^0, k^0, p^0\right) \gg \left(c^*, I^*, k^*, p^*\right)$$

so that, in particular, the steady state equilibrium capital stock is higher than the steady state optimal capital stock. This implies, of course, that the equilibrium capital stock will be inefficiently high for all t sufficiently large.

The overaccumulation of assets is the result of a pecuniary externality. When choosing the optimal capital structure, a banker takes as given the price at which liquidated assets can be sold. An increase in the probability of default will increase the expected losses from selling assets in a fire sale and reduce the expected gains from buying assets in a fire sale, thus increasing the net loss from fire sales. A central planner, unlike the banker, is not a price taker. In equilibrium, the expected cost of fire sales is equal to the expected capital gains from fire sales. Increasing the leverage of all banks reduces the price of liquidated capital goods, so the expected losses equal the expected capital gains. The planner takes this into account and chooses a leverage ratio higher than in laisser faire equilibrium. The increase in the supply of deposits raises the interest rate, lowers the price of deposits, and reduces the market value of the securities issued by banks. This in turn will reduce the price of capital goods, so fewer capital goods are produced, and the growth rate falls.

6 Conclusion

We have analyzed a model of general equilibrium in which bank capital is "expensive." Deposits earn a liquidity premium that allows banks to pay a lower return on deposits than it pays on equity. In equilibrium, there must be an offsetting cost, otherwise banks would be funded entirely by deposits. The cost of using deposits arises from the risk of default, after which banks are forced to liquidate assets in a "fire sale." A fire sale represents a transfer of value from bank creditors to purchasers, but it is not a true economic cost. The fire sale represents a pecuniary externality that leads to overaccumulation of capital. To remove the distortion in the perceived cost of funding, the regulator needs to increase the *social* cost of deposits as a source of funding. This can be done by *reducing* the level of bank capital and forcing banks to rely more heavily on deposits. The greater the supply of deposits, the lower the liquidity premium and the higher the interest rate.

The model we have analyzed is very simple. We have excluded deadweight costs of default, for example. Deadweight costs of default that are internalized by the bank, would also tend to reduce the use of deposits, but would not change the qualitative welfare results in this paper. External costs that are *ignored* by the bankers, on the other hand, would offset the distortion caused by fire sales and might reverse our welfare conclusions. Whether the pecuniary externality caused by fire sales is greater than the external costs of financial distress is, of course, an empirical question.

Another limitation of the model is that it combines the corporate and financial sectors. A model in which banks lend to firms that in turn make investments in real assets would be more realistic and might lead to different results. Gornall and Strebulaev (2015) undertake a quantitative analysis of capital structures in the banking and corporate sectors. Gale and Gottardi (2016) analyze the privately and socially optimal capital structures in a general equilibrium model with separate banking and corporate sectors.

We have ignored problems of corporate governance and, in particular, the large literature on risk shifting and asset substitution. It is worth noting that this literature assumes that banks are operated in the interest of shareholders. In other words, it deals with shareholder-debtholder conflict, but not with manager-shareholder conflict. When we take seriously the separation of ownership and control, it is not entirely clear that increasing bank equity will solve all incentive problems.

The lesson of this paper is that general equilibrium effects and pecuniary externalities matter, but there are other important questions. Much remains to be done.

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7 Appendix

In this section we present proofs of the various propositions stated in the main section of the paper. For completeness, we include in Section 7.2 a number of well known results on the efficient growth path of an economy that are used both in the proof of the planner's problem and in the proofs of some other results. The reader who is familiar with these results can skip over parts of this section.

7.1 Proof of Proposition 1

Dropping the unnecessary time subscripts, the banker's objective function can be written

$$\beta^t u\left(c\right)\left[A\int^z \theta dF + Az\left(1-F\left(z\right)\right)\right]+\psi\left[q\left(1-\delta\right)F\left(z\right)+\frac{pA\int_z\left(\theta-z\right)dF}{q}+p\left(1-\delta\right)\left(1-F\left(z\right)\right)\right].$$

The first-order condition for a maximum is obtained by direct calculation:

$$\beta^t u'(c)\left[AzF'(z) + A(1-F(z)) - AzF'(z)\right] + \\ \psi\left[q(1-\delta)F'(z) - \frac{pA(1-F(z))}{q} - p(1-\delta)F'(z)\right] = 0.$$

This can be simplified to

$$\beta^t u'(c)A(1-F(z)) + \psi\left[(q-p)(1-\delta)F'(z) - \frac{pA(1-F(z))}{q}\right] = 0$$

and rearranged to give

$$\beta^t u'(c)A + \psi\left[(q-p)(1-\delta)\frac{F'(z)}{1-F(z)} - \frac{pA}{q}\right] = 0.$$

Since ψ is positive and $(q-p)(1-\delta)$ is negative and the hazard rate $F'(z)(1-F(z))^{-1}$ is increasing in z, the expression on the left hand side is decreasing in z and therefore has at most one solution $0 < z < \Theta$.

7.2 Proof of Theorem 2

7.2.1 Maintained assumptions

The utility function satisfies the properties:

- $u : \mathbf{R}_+ \to \mathbf{R}_+$ is a C^1 function on $(0, \infty)$, $u'(c) \to \infty$ as $c \searrow 0$, and u is increasing and strictly concave.

The production function φ satisfies the properties:

- $\varphi : \mathbf{R}_+ \to \mathbf{R}_+$ is a C^1 function on $(0, \infty)$, $\varphi'(I) \to \infty$ as $I \searrow 0$ and u is increasing, and strictly concave.

The parameters A, β, δ, k_0 and φ satisfy the properties:

- $0 < A$, $0 < \beta < 1$, $0 < \delta < 1$, $k_0 > 0$.

Finally,

- there exists a number $0 < \hat{k} < \infty$ such that

$$\varphi\left(A\hat{k}\right) = \delta\hat{k},$$

that is, when the capital stock reaches the level $\hat{k}$, the entire output of the economy must be re-invested in order to replace the depreciation of the capital stock.

Proposition 9 *At any feasible solution* $\{c_t, k_t, I_t\}_{t=0}^{\infty}$*, we have* $\limsup_{t\to\infty} k_t \leq \hat{k}$*.*

Proof. Since φ is strictly concave, $\varphi(Ak) < \delta k$ for any $k > \hat{k}$. Consequently, $k_t > \hat{k}$ implies that

$$\begin{aligned} k_{t+1} &= (1-\delta) k_t + \varphi(I_t) \\ &\leq (1-\delta) k_t + \varphi(Ak_t) \\ &< (1-\delta) k_t + \delta k_t = k_t. \end{aligned}$$

If $\limsup_{t\to\infty} k_t > \hat{k}$, then $k_t \to \bar{k} > \hat{k}$. But this implies that

$$k_{t+1} - k_t \leq (1-\delta) k_t + \varphi(Ak_t) - k_t \to (1-\delta)\bar{k} + \varphi\left(A\bar{k}\right) - \bar{k} < 0,$$

contradicting $k_t \to \bar{k}$. ■

As a corollary, $A\hat{k}$ is an upper bound on the levels of consumption and investment that can be maintained indefinitely:

$$\limsup_{t\to\infty} c_t \leq A\hat{k}, \ \limsup_{t\to\infty} I_t \leq A\hat{k}.$$

7.2.2 Existence and uniqueness

The set of feasible solutions consists of sequences $\{(c_t, k_t, I_t)\}_{t=0}^{\infty}$ satisfying $(c_t, k_t, I_t) \in \mathbf{R}_+^3$,

$$c_t + I_t \leq Ak_t,$$

$$k_{t+1} \leq (1-\delta) k_t + \varphi(I_t),$$

for any $t = 1, \ldots$ and $k_0 > 0$. The feasible set is clearly non-empty, convex and compact (in the usual product topology). The objective function $\sum_{t=0}^{\infty} \beta^t u(c_t)$ is continuous (in the usual product topology) and hence attains a maximum on the feasible set. It is easy to see that the optimum is greater than $-\infty$ and this implies that (c_t, k_t, I_t) is strictly positive at each date t.[24] Finally, the strict concavity of the objective function implies that the optimal solution is unique.

Proposition 10 *Under the maintained assumptions, there is a unique solution, denoted by* $\{(c_t^*, k_t^*, I_t^*)\}_{t=0}^{\infty}$*, of the problem*

$$\begin{array}{ll} \max & \sum_t \beta^t u(c_t) \\ s.t. & c_t + I_t = Ak_t, \ \forall t \geq 0, \\ & k_{t+1} = (1-\delta) k_t + \varphi(I_t), \ \forall t \geq 0, \\ & k_0 = \bar{k}_0. \end{array}$$

For every value of t*, the solution is positive, that is,* $c_t^* > 0$*,* $k_t^* > 0$*, and* $I_t^* > 0$*.*

[24] Note first that $\varphi(I_t) \geq 0$ and $0 < \delta < 1$ implies that $k_t > 0$ for all t. Then let t_0 be the first date at which $c_{t_0} > 0$ and $c_{t_0+1} = 0$. Then $u'(0) = \infty$ implies that for some $\varepsilon > 0$, reducing consumption at t_0 by ε and increasing consumption at $t_0 + 1$ by $A\varphi'(I_{t_0})\varepsilon$ must increase the objective function. A similar argument shows that there cannot be a date t_0 such that $I_{t_0} > 0$ and $I_{t_0+1} = 0$, because $\varphi'(0) = \infty$. Thus, $c_t, I_t, k_t > 0$ for all t.

Let $\mathcal{V}$ denote the set of functions $V : [0, \hat{k}] \to \left[0, \frac{u(A\hat{k})}{1-\beta}\right]$ that are continuous, increasing, and strictly concave. Then define the transformation $T : \mathcal{V} \to \mathcal{V}$ by putting

$$TV(k) = \sup\{u(c') + \beta V(k') : c' + I' = Ak,\ k' = (1-\delta)k + \varphi(I')\},$$

for any $k \in [0, \hat{k}]$. It is clear that $V \in \mathcal{V}$ implies that $TV \in \mathcal{V}$, so the transformation T is well defined. For any two functions $V_1, V_2 \in \mathcal{V}$,

$$\sup_{0 \le k \le \hat{k}} |TV_1(k) - TV_2(k)| \le \beta \sup_{0 \le k \le \hat{k}} |V_1(k) - V_2(k)|$$

so there exists a fixed point $TV^* = V^*$ by the Banach fixed point theorem.

The function V is C^1 on $\left(0, \hat{k}\right)$ as shown by Benveniste and Scheinkman (1979).

7.2.3 Kuhn-Tucker conditions

The Kuhn-Tucker Theorem tells us that if $\{(c_t^*, I_t^*, k_t^*)\}_{t=1}^{\infty}$ solves the problem

$$\begin{array}{ll} \min & -\sum_{t=1}^{\infty} \beta^t u(c_t) \\ \text{s.t.} & c_t + I_t - Ak_t \le 0,\ \forall t \ge 0, \\ & k_{t+1} - (1-\delta)k_t - \varphi(I_t) \le 0,\ \forall t \ge 0, \\ & k_1 = k_0 \end{array}$$

and the constraint qualification is satisfied, then there exist non-negative scalars $\{(\alpha_t^*, \gamma_t^*)\}_{t=1}^{\infty}$ such that

$$\begin{aligned} -\beta^t u'(c_t^*) + \alpha_t^* &\le 0,\ \forall t \ge 0, \\ \alpha_t^* - \gamma_t^* \varphi'(I_t^*) &\le 0,\ \forall t \ge 0, \\ -\alpha_{t+1}^* A + \gamma_t^* - \gamma_{t+1}^*(1-\delta) &\le 0,\ \forall t \ge 0. \end{aligned}$$

and

$$\begin{aligned} \alpha_t^*(c_t^* + I_t^* - Ak_t^*) &= 0,\ \forall t \ge 0, \\ \gamma_t^*\left(k_{t+1}^* - (1-\delta)k_t^* - \varphi(I_t^*)\right) &= 0,\ \forall t \ge 0. \end{aligned}$$

At an interior solution, where $(c_t^*, I_t^*, k_t^*) \gg \mathbf{0}$ for every t, the first-order conditions for minimizing the Lagrangean are

$$\begin{aligned} -\beta^t u'(c_t^*) + \alpha_t^* &= 0,\ \forall t \ge 0, \\ \alpha_t^* - \gamma_t^* \varphi'(I_t^*) &= 0,\ \forall t \ge 0, \\ -\alpha_{t+1}^* A + \gamma_t^* - \gamma_{t+1}^*(1-\delta) &= 0,\ \forall t \ge 0. \end{aligned}$$

In the case of a convex programming problem, the first-order conditions are also sufficient. Suppose that $\{(c_t^*, I_t^*, k_t^*)\}_{t=1}^{\infty}$ is a feasible solution of the planner's problem that satisfies the first-order conditions and $(c_t^*, I_t^*, k_t^*) \gg \mathbf{0}$ for every t and let $\{(c_t, I_t, k_t)\}_{t=1}^{\infty}$ be an arbitrary feasible solution. The Lagrangean

$$-\sum_{t=1}^{\infty} \left\{\beta^t u(c_t) + \alpha_t^* (c_t + I_t - Ak_t) + \gamma_t^* (k_{t+1} - (1-\delta) k_t - \varphi(I_t))\right\}$$

is a convex function and so the first-order conditions are sufficient for a minimum. Then

$$\begin{aligned} -\sum_{t=1}^{\infty} \beta^t u(c_t^*) &= -\sum_{t=1}^{\infty} \left\{\beta^t u(c_t^*) + \alpha_t^* (c_t^* + I_t^* - Ak_t^*) + \gamma_t^* \left(k_{t+1}^* - (1-\delta) k_t^* - \varphi(I_t^*)\right)\right\} \\ &\leq -\sum_{t=1}^{\infty} \left\{\beta^t u(c_t) + \alpha_t^* (c_t + I_t - Ak_t) + \gamma_t^* (k_{t+1} - (1-\delta) k_t - \varphi(I_t))\right\} \\ &\leq -\sum_{t=1}^{\infty} \beta^t u(c_t), \end{aligned}$$

where the first line follows because the complementary slackness conditions are satisfied, the second follows because $\{(c_t^*, I_t^*, k_t^*)\}_{t=1}^{\infty}$ minimizes the Lagrangean and the last follows from the feasibility of $\{(c_t, I_t, k_t)\}_{t=1}^{\infty}$.

7.2.4 Convergence

A feasible solution $\{(c_t^*, k_t^*, I_t^*)\}_{t=0}^{\infty}$ is a steady-state if

$$(c_t^*, k_t^*, I_t^*) = (c^*, k^*, I^*), \ \forall t = 0, 1, ...$$

Then feasibility implies that $k_0 = k^*$ and

$$k^* = (1-\delta) k^* + \varphi(I^*),$$

or $\varphi(I^*) = \delta k^*$.

The first-order conditions for the steady state are:

$$u'(c^*) = \beta V'(k^*) \varphi'(I^*)$$

and

$$V'(k^*) = u'(c^*) A + \beta V'(k^*)(1-\delta)$$

which can be combined to yield

$$V'(k^*) = \beta V'(k^*) \varphi'(I^*) A + \beta V'(k^*)(1-\delta).$$

Dividing by $V'(k^*)$ we obtain

$$\frac{1}{\varphi'(I^*)} = \frac{\beta A}{1-\beta(1-\delta)}.$$

Similarly, the first-order conditions for a non-steady-state equilibrium are

$$u'(c_t) = \beta V'(k_{t+1})\,\varphi'(I_t)$$

and

$$V'(k_t) = u'(c_t)\,A + \beta V'(k_{t+1})\,(1-\delta)\,,$$

which can be combined to yield

$$V'(k_t) = \beta V'(k_{t+1})\,\varphi'(I_t)\,A + \beta V'(k_{t+1})\,(1-\delta)\,.$$

Now let us assume that $k_t < k^*$ and, contrary to what we want to prove, $k_{t+1} \leq k_t$, so that $V'(k_t) \leq V'(k_{t+1})$. Then, dividing by $V'(k_{t+1})$, we get

$$1 \geq \frac{V'(k_t)}{V'(k_{t+1})} = \beta\varphi'(I_t)\,A + \beta\,(1-\delta)\,,$$

or

$$\frac{1}{\varphi'(I_t)} \geq \frac{\beta A}{1-\beta\,(1-\delta)}.$$

Comparing this with the steady state analogue, we see that $\varphi'(I_t) \leq \varphi'(I^*)$, so that $I_t \geq I^*$. But this implies that

$$\varphi(I_t) \geq \varphi(I^*) = \delta k^* > \delta k_t,$$

so

$$\begin{aligned} k_{t+1} &\geq (1-\delta)\,k_t + \varphi(I_t) \\ &> (1-\delta)\,k_t + \delta k_t = k_t, \end{aligned}$$

contradicting our assumption. Thus, $k_{t+1} > k_t$.

Now suppose that $k_{t+1} \geq k^* > k_t$. Then $I_t > I^*$, which implies $\varphi'(I_t) < \varphi'(I^*)$ and $c_t < c^*$, so that from the steady-state first-order condition

$$\begin{aligned} u'(c_t) &> u'(c^*) = \beta V'(k^*)\,\varphi'(I^*) \\ &> \beta V'(k_{t+1})\,\varphi'(I_t)\,, \end{aligned}$$

because $V'(k^*) \geq V'(k_{t+1})$. This contradicts the first-order condition for the non-steady-state equilibrium, and shows that $k_t < k_{t+1} < k^*$.

A similar argument shows that if $k_t > k^*$ then $k_t > k_{t+1} > k^*$. It is easy to see that, from any initial condition $k_0 \neq k^*$, the optimal path will converge monotonically to a steady state and this steady state must be the unique steady-state optimum.

7.3 Proof of Theorem 3

The proof is constructive and proceeds in a number of steps.

Step 1 We begin by setting $(c_t, k_t, I_t) = (c_t^*, k_t^*, I_t^*)$ for every date t. For each t, choose z_t^* so that equation (7) is satisfied. This is clearly possible since $0 \leq c_t^* \leq Ak_t^*$ and the right hand side of equation (7) is continuous in z_t^* and varies from 0 to Ak_t^* as z_t^* varies from 0 to Θ.

Step 2 Next, we can use equation (11) to define q_t for every t.

Step 3 The first feasibility condition for the planner's problem requires $c_t^* + I_t^* = Ak_t^*$ for every t. Therefore, feasibility and equation (7) imply that

$$\begin{aligned} I_t^* &= Ak_t^* - c_t^* \\ &= Ak_t^* - Ak_t^* \int^{z_t^*} \theta dF - Az_t^* k_t \left(1 - F\left(z_t^*\right)\right) \\ &= Ak_t^* \int_{z_t^*} \theta dF - Az_t^* k_t \left(1 - F\left(z_t^*\right)\right) \\ &= Ak_t^* \int_{z_t^*} \left(\theta - z_t^*\right) dF, \end{aligned}$$

as required by equation (8).

Step 4 Set $\psi_t = \beta^t u'\left(c_t^*\right)$ and $p_t = \frac{\gamma_t^*}{\alpha_t^*}$ for every t. Then use equation (4) to define r_t

$$\begin{aligned} r_{t-1} &= \frac{\beta^t u'\left(c_t\right)}{\beta^{t-1} u'\left(c_{t-1}\right)} \left(\lambda_t + \mu_t\right) \\ &= \frac{\beta u'\left(c_t\right)}{u'\left(c_{t-1}\right)} \left(\lambda_t + \mu_t\right), \end{aligned}$$

for every t, and use equation (5) to define v_t

$$\begin{aligned} v_{t-1} &= \frac{\beta^t u'\left(c_t\right)}{\beta^{t-1} u'\left(c_{t-1}\right)} R_t \\ &= \frac{\beta u'\left(c_t\right)}{u'\left(c_{t-1}\right)} R_t, \end{aligned}$$

for every t, where $R_t = A - z_t^* A(\lambda_t + \mu_t) + p_t(1-\delta)$. Note that consistency requires that

$$\begin{aligned}
p_{t-1} &= z_t^* A r_{t-1} + v_{t-1} \\
&= \frac{\beta u'(c_t)}{u'(c_{t-1})} z_t^* A(\lambda_t + \mu_t) + \frac{\beta u'(c_t)}{u'(c_{t-1})} R_t \\
&= \frac{\beta u'(c_t)}{u'(c_{t-1})} z_t^* A(\lambda_t + \mu_t) + \frac{\beta u'(c_t)}{u'(c_{t-1})} (A - z_t^* A(\lambda_t + \mu_t) + p_t(1-\delta)) \\
&= \frac{\beta u'(c_t)}{u'(c_{t-1})} (A + p_t(1-\delta)) \\
&= \frac{\alpha_t^*}{\alpha_{t-1}^*} (A + p_t(1-\delta)).
\end{aligned}$$

Substituting the definitions for p_{t-1} and p_t, we obtain

$$\begin{aligned}
\frac{\gamma_{t-1}^*}{\alpha_{t-1}^*} &= \frac{\alpha_t^*}{\alpha_{t-1}^*}\left(A + \frac{\gamma_t^*}{\alpha_t^*}(1-\delta)\right) \\
\iff \gamma_{t-1}^* &= \alpha_t^* A + \gamma_t^*(1-\delta),
\end{aligned}$$

which is precisely the first-order condition in the planner's problem. Thus, our definitions are consistent.

Step 5 Finally, we have to check that the first-order condition for profit maximization in the capital goods industry, equation (2), is satisfied. But the first-order condition for an interior solution to the planner's problem guarantees that

$$\frac{\gamma_t^*}{\alpha_t^*}\varphi'(I_t^*) = 1,$$

which implies (2) when $p_t = \frac{\gamma_t^*}{\alpha_t^*}$.

7.4 Proof of Proposition 4

$(i) \implies (ii)$ The first-order condition for the choice of z_{t+1}^* is

$$\beta^{t+1} u'\left(c_{t+1}^*\right) A + \psi_{t+1}^* \left(\left(q_{t+1}^* - p_{t+1}^*\right)(1-\delta) \frac{F'\left(z_{t+1}^*\right)}{1 - F\left(z_{t+1}^*\right)} - \frac{p_{t+1}^* A}{q_{t+1}^*} \right) = 0.$$

If $\beta^{t+1} u'\left(c_{t+1}^*\right) A = \psi_{t+1}^*$, the first-order condition can be written as

$$A + \left(q_{t+1}^* - p_{t+1}^*\right)(1-\delta) \frac{F'\left(z_{t+1}^*\right)}{1 - F\left(z_{t+1}^*\right)} - \frac{p_{t+1}^* A}{q_{t+1}^*} = 0.$$

If $q_{t+1}^* \neq p_{t+1}^*$, the first-order condition is equivalent to

$$\frac{A}{q_{t+1}^*} + (1-\delta) \frac{F'\left(z_{t+1}^*\right)}{1 - F\left(z_{t+1}^*\right)} = 0,$$

which is impossible, so $\beta^{t+1} u'\left(c_{t+1}^*\right) A = \psi_{t+1}^*$ implies that $q_{t+1}^* = p_{t+1}^*$.

$(ii) \implies (i)$ Conversely, if $q_{t+1}^* = p_{t+1}^*$, then the first-order condition becomes

$$\beta^{t+1} u' \left(c_{t+1}^*\right) A - \psi_{t+1}^* A = 0,$$

which implies that $\beta^{t+1} u' \left(c_{t+1}^*\right) = \psi_{t+1}^*$.

$(ii) \iff (iii)$ The market-clearing conditions

$$q_t^* = \min \left\{ \frac{A \int_{z_t^*} (\theta_t - z_t^*) \, dF}{(1-\delta) F(z_t^*)}, p_t^* \right\}$$

and

$$I_t^* = A k_t^* \int_{z_t^*} (\theta_t - z_t^*) \, dF$$

imply that

$$q_t^* = \min \left\{ \frac{I_t^*}{(1-\delta) F(z_t^*) k_t^*}, p_t^* \right\},$$

from which it is immediately clear that $q_t^* = p_t^*$ if and only if

$$p_t^* (1-\delta) F(z_t^*) k_t^* \leq I_t^*.$$

This completes the proof that the three conditions are equivalent.

7.5 Proof of Theorem 6

Proof. The proof is by contradiction. Suppose that $\{(c_t^*, d_t^*, e_t^*, I_t^*, k_t^*, z_t^*)\}_{t=1}^{\infty}$ is Pareto efficient and, contrary to what we want to prove, $\psi_t^* \neq \beta^t u(c_t^*)$ for some t. Suppose first that $\psi_t^* > \beta^t u(c_t^*)$ for some date t. The first-order condition for z_{t+1},

$$\beta^{t+1} u'(c_{t+1}) A + \psi_{t+1} \left((q_{t+1} - p_{t+1})(1-\delta) \frac{F'(z_{t+1})}{1 - F(z_{t+1})} - \frac{p_{t+1} A}{q_{t+1}} \right) = 0,$$

implies that

$$\begin{aligned} A + \left((q_{t+1} - p_{t+1})(1-\delta) \frac{F'(z_{t+1})}{1 - F(z_{t+1})} - \frac{p_{t+1} A}{q_{t+1}} \right) &> 0, \\ \implies A \left(\frac{q_{t+1} - p_{t+1}}{q_{t+1}} \right) &> (p_{t+1} - q_{t+1})(1-\delta) \frac{F'(z_{t+1})}{1 - F(z_{t+1})} \\ \implies -\frac{A}{q_{t+1}} &> (1-\delta) \frac{F'(z_{t+1})}{1 - F(z_{t+1})}, \end{aligned}$$

a contradiction. Thus, it must be the case in any equilibrium that $\psi_{t+1} \leq \beta^{t+1} u(c_{t+1})$ for all t.

Suppose then, that the equilibrium allocation is efficient and that $\psi_{t+1} < \beta^{t+1} u(c_{t+1})$, for some t. Since the equilibrium allocation is determined by the sequence of investments $\{I_t^*\}_{t=1}^{\infty}$, efficiency requires that $p_t^* = \frac{\gamma_t^*}{\alpha_t^*}$, for all t, where α_t^* and γ_t^* are the multipliers from the planner's problem. A planner's solution must satisfy the first-order condition

$$\alpha_{t+1}^* A + \gamma_{t+1}^* (1-\delta) = \gamma_t^*$$

for every date t. This equation is equivalent to

$$\frac{\alpha_{t+1}^*}{\alpha_t^*} \left(A + p_{t+1}^* (1-\delta)\right) = p_t^*, \tag{30}$$

since $\gamma_t^* = \alpha_t^* p_t^*$ and $\gamma_{t+1}^* = \alpha_{t+1}^* p_{t+1}^*$, according to the first-order conditions.

In equilibrium, the price of one unit of capital goods must equal the market value of a firm with one unit of capital. Substituting for prices from the first-order conditions in the consumer's problem, we obtain the following condition:

$$\begin{aligned}
p_t^* &= v_t^* + r_t d_{t+1} \\
&= \frac{\psi_{t+1}^*}{\psi_t^*} R_{t+1} + \frac{\beta^{t+1} u'\left(c_{t+1}^*\right)}{\psi_t^*} \lambda_{t+1} d_{t+1}^* + \frac{\psi_{t+1}^*}{\psi_t^*} \mu_{t+1} d_{t+1}^* \\
&= \frac{\beta^{t+1} u'\left(c_{t+1}^*\right)}{\psi_t^*} \lambda_{t+1} d_{t+1}^* + \frac{\psi_{t+1}^*}{\psi_t^*} \left(R_{t+1} + \mu_{t+1} d_{t+1}^*\right) \\
&= \frac{\psi_{t+1}^*}{\psi_t^*} \left\{\lambda_{t+1} d_{t+1}^* + \left(A - \lambda_{t+1} d_{t+1}^* + p_{t+1}^* (1-\delta)\right)\right\} + \frac{\beta^{t+1} u'\left(c_{t+1}^*\right) - \psi_{t+1}^*}{\psi_t^*} \lambda_{t+1} d_{t+1}^* \\
&= \frac{\psi_{t+1}^*}{\psi_t^*} \left(A + p_{t+1}^* (1-\delta)\right) + \frac{\beta^{t+1} u'\left(c_{t+1}^*\right) - \psi_{t+1}^*}{\psi_t^*} \lambda_{t+1} d_{t+1}^*. \qquad (31)
\end{aligned}$$

In the steady state to which the efficient equilibrium path converges, the first-order condition for efficiency becomes

$$p^* = A + p^* (1-\delta)$$

and

$$\lim_{t\to\infty} \frac{\psi_{t+1}^*}{\psi_t^*} = 1.$$

Then the equilibrium first-order condition implies that

$$\lim_{t\to\infty} \frac{\beta^{t+1} u'\left(c_{t+1}^*\right) - \psi_{t+1}^*}{\psi_t^*} \lambda_{t+1} d_{t+1}^* = 0. \tag{32}$$

Comparing equations (30) and (31) above, it is clear that

$$\frac{\psi_{t+1}^*}{\psi_t^*} \left(A + p_{t+1}^* (1-\delta)\right) \leq p_t^* = \frac{\alpha_{t+1}^*}{\alpha_t^*} \left(A + p_{t+1}^* (1-\delta)\right),$$

or

$$\frac{\psi^*_{t+1}}{\psi^*_t} \leq \frac{\alpha^*_{t+1}}{\alpha^*_t},$$

for any t. Suppose that $\psi^*_t = \alpha^*_t$ for $t = 1, ..., \bar{t}$ and $\psi^*_{\bar{t}+1} < \alpha^*_{\bar{t}+1}(1-\varepsilon)$. We now claim that $\psi^*_t \leq \alpha^*_t(1-\eta)$ for all $t > \bar{t}+1$. The proof is by induction. Suppose that for some k and all $i = 2, ..., k$,

$$\psi^*_{\bar{t}+i} \leq \alpha^*_{\bar{t}+i}(1-\eta).$$

Then,

$$\begin{aligned}
\psi^*_{\bar{t}+k+1} &\leq \alpha^*_{\bar{t}+k+1}\frac{\psi^*_{\bar{t}+k}}{\alpha^*_{\bar{t}+k}} \\
&\leq \alpha^*_{\bar{t}+k+1}\frac{\alpha^*_{\bar{t}+k}(1-\eta)}{\alpha^*_{\bar{t}+k}} \\
&= \alpha^*_{\bar{t}+k+1}(1-\eta).
\end{aligned}$$

Thus, by induction,

$$\psi^*_t \leq \alpha^*_t(1-\eta)$$

for all $t > \bar{t}+1$. Then

$$\begin{aligned}
\lim_{t\to\infty}\frac{\beta^{t+1}u'\left(c^*_{t+1}\right) - \psi^*_{t+1}}{\psi^*_t} &= \lim_{t\to\infty}\frac{\alpha^*_{t+1} - \psi^*_{t+1}}{\psi^*_t} \\
&\geq \lim_{t\to\infty}\frac{\alpha^*_{t+1}\eta}{\alpha^*_t(1-\eta)} = \frac{\eta}{1-\eta} > 0,
\end{aligned}$$

contradicting (32). This proves that the conditions in Proposition 4 must be satisfied for every date t in order to have an efficient equilibrium. ■

7.6 Proof of Theorem 7 and Proposition 8

Step 1 From equation (27) and the Implicit Function Theorem, we can write investment as a C^1 and increasing function $I(p)$. Then equation (29) implies that the ratio of investment to capital is a C^1 and increasing function

$$\rho_I(p) \equiv \frac{\delta I(p)}{\varphi(I(p))}.$$

Then equation (28) implies that the ratio of consumption to capital is a C^1 and decreasing function

$$\rho_c(p) = A - \rho_I(p).$$

We can rewrite the equation (26) as

$$\begin{aligned}
\frac{(1-\beta(1-\delta))}{\beta}\psi p &= \frac{c}{k} + \psi\frac{I}{k} \\
&= \rho_c(p) + \psi\rho_I(p).
\end{aligned}$$

Step 2 The next step is to show that the equation

$$\frac{(1-\beta(1-\delta))}{\beta}\psi p-\rho_c(p)-\psi\rho_I(p)=0$$

satisfies the conditions of the Implicit Function Theorem. Direct calculation shows that

$$\begin{aligned}
\frac{\partial}{\partial p}\left(\frac{(1-\beta(1-\delta))}{\beta}\psi p-\rho_c(p)-\psi\rho_I(p)\right) &= \frac{(1-\beta(1-\delta))}{\beta}\psi-\rho_c'(p)-\psi\rho_I'(p)\\
&\geq \frac{(1-\beta(1-\delta))}{\beta}\psi-\rho_c'(p)-\rho_I'(p)\\
&= \frac{(1-\beta(1-\delta))}{\beta}\psi>0
\end{aligned}$$

and

$$\frac{\partial}{\partial \psi}\left(\frac{(1-\beta(1-\delta))}{\beta}\psi p-\rho_c(p)-\psi\rho_I(p)\right)=\frac{(1-\beta(1-\delta))}{\beta}p-\rho_I(p).$$

Then p is a C^1 function of ψ and

$$\left(\frac{(1-\beta(1-\delta))}{\beta}\psi-\rho_c'(p)-\psi\rho_I'(p)\right)\frac{dp}{d\psi}+\left(\frac{(1-\beta(1-\delta))}{\beta}p-\rho_I(p)\right)=0,$$

or

$$\frac{dp}{d\psi}=\frac{-((1-\beta(1-\delta))p-\beta\rho_I(p))}{((1-\beta(1-\delta))\psi-\beta\rho_c'(p)-\beta\psi\rho_I'(p))}.$$

On the right hand side, we have already noted that the denominator is positive. The numerator, on the other hand, is negative. From equation (26), we have

$$\begin{aligned}
(1-\beta(1-\delta))p &= \frac{\beta(c+\psi I)}{\psi k}\\
&> \frac{\beta(\psi c+\psi I)}{\psi k},\text{ since }\psi<1,\\
&= \frac{\beta(c+I)}{k}>\frac{\beta I}{k}.
\end{aligned}$$

Then

$$(1-\beta(1-\delta))p-\beta\rho_I(p)>0.$$

It follows that p is decreasing in ψ.

Step 3 We have now established the following facts.

1. For any $\psi\in[\psi^*,1]$, there is a unique value of $p=\Phi_p(\psi)$ such that (p,ψ) satisfies equation (26), where Φ_p is a C^1 and decreasing function.

2. For any $\psi\in[\psi^*,1]$, there are unique values of $c=\Phi_c(\psi)$, $I=\Phi_I(\psi)$, and $k=\Phi_k(\psi)$ such that (I,p) satisfies (27), (c,I,k) satisfies (28), and (I,k) satisfies (29), where Φ_c, Φ_I, and Φ_k are C^1 functions and Φ_I, and Φ_k are decreasing.

This completes the proof of the two results.