Approximate controllability of Lagrangian trajectories of the 3D Navier–Stokes system by a finite-dimensional force

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Abstract

In the Eulerian approach, the motion of an incompressible fluid is usually described by the velocity field which is given by the Navier–Stokes system. The velocity field generates a flow in the space of volume-preserving diffeomorphisms. The latter plays a central role in the Lagrangian description of a fluid, since it allows to identify the trajectories of individual particles. In this paper, we show that the velocity field of the fluid and the corresponding flow of diffeomorphisms can be simultaneously approximately controlled using a finite-dimensional external force. The proof is based on some methods from the geometric control theory introduced by Agrachev and Sarychev.

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0 Introduction

The motion of an incompressible fluid is described by the following Navier–Stokes (NS) system

$$\dot{u} - \nu\Delta u + \langle u, \nabla\rangle u + \nabla p = f(t,x), \quad \operatorname{div} u = 0, \tag{0.1}$$

$$u(0) = u_0, \tag{0.2}$$

where $\nu > 0$ is the kinematic viscosity, $u = (u_1(t,x), u_2(t,x), u_3(t,x))$ is the velocity field of the fluid, $p = p(t,x)$ is the pressure, and f is an external force. Throughout this paper, we shall assume that the space variable $x = (x_1, x_2, x_3)$ belongs to the torus $\mathbb{T}^3 = \mathbb{R}^3/2\pi\mathbb{Z}^3$.

The well-posedness of the 3D NS system (0.1) is a famous open problem. Given a smooth data (u_0, f), the existence and uniqueness of a smooth solution is known to hold only locally in time. One can establish global existence in the case of a small data. Global existence for a large data holds in the case of weak solution, but in that case the uniqueness is open.

The flow generated by a sufficiently smooth velocity field u gives the Lagrangian trajectories of the fluid:

$$\dot{x} = u(t,x), \quad x(0) = x_0 \in \mathbb{T}^3. \tag{0.3}$$

Since the fluid is assumed to be incompressible, for any $t \geq 0$, the mapping $\phi_t^u : x_0 \mapsto x(t)$ belongs to the group $\operatorname{SDiff}(\mathbb{T}^3)$ of orientation and volume preserving diffeomorphisms on $\mathbb{T}^3$ isotopic to the identity. This group is often referred as *configuration space* of the fluid (cf. [AK98, KW09]). Thus for a sufficiently smooth data, we have a path $(u(t), \phi_t^u)$, which is defined locally in time, and its approximate controllability is the main issue addressed in this paper. We shall assume that the external force is of the following form

$$f(t,x) = h(t,x) + \eta(t,x),$$

where h is the fixed part of the force (given function) and η is a control force. To state the main result of this paper, we need to introduce some notation. Let us define the space

$$H := \{u \in L^2(\mathbb{T}^3, \mathbb{R}^3) : \operatorname{div} u = 0, \quad \int_{\mathbb{T}^3} u(x)\mathrm{d}x = 0\}, \tag{0.4}$$

and denote by Π the orthogonal projection onto H in $L^2(\mathbb{T}^3, \mathbb{R}^3)$. Consider the projection of system (0.1) onto H:

$$\dot{u} + Lu + B(u) = h(t,x) + \eta(t,x), \tag{0.5}$$

where $L = -\Delta$ is the Stokes operator and $B(u) := \Pi(\langle u, \nabla\rangle u)$. Let us set $H^k_\sigma := H^k(\mathbb{T}^3, \mathbb{R}^3) \cap H$, where $H^k(\mathbb{T}^3, \mathbb{R}^3)$ is the space of vector functions $v = (v_1, v_2, v_3)$ with components in the usual Sobolev space of order k on $\mathbb{T}^3$. Let E be subset of H. We shall say that system (0.5) is approximately controllable by an E-valued control, if for any $\nu > 0$, $k \geq 3$, $\varepsilon > 0$, $T > 0$, $u_0, u_1 \in H^k_\sigma$, $h \in L^2(J_T, H^{k-1}_\sigma)$, and $\psi \in \mathrm{SDiff}(\mathbb{T}^3)$, there is a control $\eta \in L^2([0,T], E)$ and a solution u of (0.5), (0.2) satisfying

$$\|u(T) - u_1\|_{H^k(\mathbb{T}^3)} + \|\phi^u_T - \psi\|_{C^1(\mathbb{T}^3)} < \varepsilon.$$

The following theorem is a simplified version of our main result (see Section 2.1).

Main Theorem. *There is a finite-dimensional subspace $E \subset H$ such that (0.5) is approximately controllable by an E-valued control.*

Roughly speaking, this shows that, using a finite-dimensional external force, one can drive the fluid flow (which starts at the identity) arbitrarily close to any configuration $\psi \in \mathrm{SDiff}(\mathbb{T}^3)$. Moreover, near the final position $\psi(x)$, the particle starting from x will have approximately the prescribed velocity $v_1(x) := u_1(\psi(x))$.

We give some explicit examples of finite-dimensional subspaces E which ensure the above approximate controllability property. For instance, for any $\ell \in \mathbb{Z}^3$, let $\{l(\ell), l(-\ell)\}$ be an arbitrary orthonormal basis in $\{x \in \mathbb{R}^3 : \langle x, \ell\rangle = 0\}$. We show that our problem is controllable by η taking values in a space of the form

$$E = E(\mathcal{K}) := \mathrm{span}\{l(\pm\ell)\cos\langle \ell, x\rangle, l(\pm\ell)\sin\langle \ell, x\rangle : \ell \in \mathcal{K}\},\ \mathcal{K} \subset \mathbb{Z}^3 \qquad (0.6)$$

if and only if $\mathcal{K}$ is a generator of $\mathbb{Z}^3$ (i.e., any $a \in \mathbb{Z}^3$ is a finite linear combination of the elements of $\mathcal{K}$ with integer coefficients). The simplest example of a generator of $\mathbb{Z}^3$ is

$$\mathcal{K} = \{(1,0,0), (0,1,0), (0,0,1)\},$$

in which case $\dim E(\mathcal{K}) = 12$. We also establish approximate controllability of the system in question by controls having two vanishing components. More precisely, the space E can be chosen of the form

$$E = \Pi\{(0,0,1)\zeta : \zeta \in \mathcal{H}\}, \qquad (0.7)$$

where

$$\mathcal{H} := \mathrm{span}\{\sin\langle m, x\rangle, \cos\langle m, x\rangle : m \in \mathcal{K}\}$$

and $\mathcal{K} := \{(1,0,0), (0,1,0), (1,0,1), (0,1,1)\}$ (i.e., $\dim E = 8$). In (2.32) an example of a 6-dimensional subspace is given which guarantees the controllability of the 3D NS system.

The strategy of the proof of Main Theorem is based on some methods introduced by Agrachev and Sarychev in [AS05] and [AS06]. In that papers they prove approximate controllability for the 2D NS and Euler systems by a finite-dimensional force. This method is then developed and generalised by several authors for various PDE's. Rodrigues [Rod07] proves controllability for the 2D NS system on a rectangle with Lions boundary conditions. Shirikyan [Shi06, Shi07, Shi13] studies controllability for the 3D NS system on the torus and 1D viscous Burgers equation on the real line. The case of incompressible and compressible 3D Euler equations is considered by Nersisyan in [Ner10, Ner11], and the controllability for the 2D defocusing cubic Schrödinger equation is established by Sarychev in [Sar12].

All the above papers are concerned with the problem of controllability of the velocity field. The controllability of the Lagrangian trajectories of 2D and 3D Euler equations is studied by Glass and Horsin [GH10, GH12], in the case of boundary controls. For given two smooth contractible sets γ_1 and γ_2 of fluid particles which surround the same volume, they construct a control such that the corresponding flow drives γ_1 arbitrarily close to γ_2. In the context of our paper, a similar property can be derived from our main result. Indeed, Krygin shows in [Kry71] that there is a diffeomorphism $\psi \in \mathrm{SDiff}(\mathbb{T}^3)$ such that $\psi(\gamma_1) = \gamma_2$. Thus we can find an E-valued control η such that $\phi_T^u(\gamma_1)$ is arbitrarily close to γ_2, and, moreover, at time T the particles will have approximately the desired velocity.

When E is of the form (0.7), our Main Theorem is related to the recent paper [CL12] by Coron and Lissy. In that paper, the authors establish local null controllability of the velocity for the 3D NS system controlled by a distributed force having two vanishing components (i.e., the controls are valued in a space of the form (0.7), where $\mathcal{H}$ is the space of space-time L^2-functions supported in a given open subset). The reader is referred to the book [Cor07] for an introduction to the control theory of the NS system by distributed controls and for references on that topic.

Let us give a brief (and not completely accurate) description of how the Agrachev–Sarychev method is adapted to establish approximate controllability in the above-defined sense. We assume that E is given by (0.6) for some generator $\mathcal{K}$ of $\mathbb{Z}^3$. Let $\psi \in \mathrm{SDiff}(\mathbb{T}^3)$ and let $h(t,x)$ be a smooth isotopy connecting it to the identity: $h(0,x) = x$ and $h(T,x) = \psi(x)$. Then $\hat{u}(t,x) := \dot{h}(t, h^{-1}(t,x))$ is a divergence-free vector field such that $\phi_t^{\hat{u}}(x) = h(t,x)$ for all $t \in [0,T]$. In particular, $\phi_T^{\hat{u}} = \psi$. The mapping $u \mapsto \phi_T^u$ is continuous from $L^1([0,T], H_\sigma^k)$ to $C^1(\mathbb{T}^3)$, where $L^1([0,T], H_\sigma^k)$ is endowed with the *relaxation* norm

$$|||u|||_{T,k} := \sup_{t\in[0,T]} \left\| \int_0^t u(s)\mathrm{d}s \right\|_{H^k(\mathbb{T}^3)}.$$

Hence we can choose a smooth vector field u sufficiently close to $\hat{u}$ with respect to this norm, so that

$$u(0) = u_0, \quad u(T) = u_1, \quad \|\phi_T^u - \psi\|_{C^1(\mathbb{T}^3)} < \varepsilon.$$

Then u is a solution of our system corresponding to a control η_0, which can be explicitly expressed in terms of u and h from equation (0.5). In general, this control η_0 is not E-valued, so we need to approach u appropriately with solutions corresponding to E-valued controls. To this end, we define the sets

$$\mathcal{K}_0 := \mathcal{K}, \quad \mathcal{K}_j = \mathcal{K}_{j-1} \cup \{m \pm n : m, n \in \mathcal{K}_{j-1}\}, \quad j \geq 1.$$

As $\mathcal{K}$ is a generator of $\mathbb{Z}^3$, one easily gets that $\cup_{j\geq 1}\mathcal{K}_j = \mathbb{Z}^3$, hence $\cup_{j\geq 1}E(\mathcal{K}_j)$ is dense in H^k_σ. Let P_N be the orthogonal projection onto $E(\mathcal{K}_N)$ in H. Then a perturbative result implies that, for a sufficiently large $N \geq 1$, system (0.5), (0.2) with control $P_N\eta_0$ has a smooth solution u_N verifying

$$\|u_N(T) - u_1\|_{H^k(\mathbb{T}^3)} + \|\phi_T^{u_N} - \psi\|_{C^1(\mathbb{T}^3)} < \varepsilon.$$

On the other hand, if we consider the following auxiliary system

$$\dot{u} + \nu L(u + \zeta) + B(u + \zeta) = h + \eta \tag{0.8}$$

with two controls ζ and η, then the below two properties hold true

Convexification principle. For any $\varepsilon > 0$ and any solution u_j of (0.5), (0.2) with an $E(\mathcal{K}_j)$-valued control η_1, there are $E(\mathcal{K}_{j-1})$-valued controls ζ and η and a solution $\tilde{u}_{j-1}$ of (0.8), (0.2) such that

$$\|u_j(T) - \tilde{u}_{j-1}(T)\|_{H^k(\mathbb{T}^3)} + |||u_j - \tilde{u}_{j-1}|||_{T,k} < \varepsilon.$$

Extension principle. For any $\varepsilon > 0$ and any solution $\tilde{u}_j$ of (0.8), (0.2) with $E(\mathcal{K}_j)$-valued controls ζ and η, there is an $E(\mathcal{K}_j)$-valued control η_2 and a solution u_j of (0.5), (0.2) such that

$$\|u_j(T) - \tilde{u}_j(T)\|_{H^k(\mathbb{T}^3)} + |||u_j - \tilde{u}_j|||_{T,k} < \varepsilon.$$

These two principles and the above-mentioned continuity property of ϕ_T^u with respect to the relaxation norm imply that, for any solution u_j of (0.5), (0.2) with an $E(\mathcal{K}_j)$-valued control η_1, there is an $E(\mathcal{K}_{j-1})$-valued control η_2 and a solution u_{j-1} of (0.5), (0.2) such that

$$\|u_j(T) - \tilde{u}_{j-1}(T)\|_{H^k(\mathbb{T}^3)} + \|\phi_T^{u_j} - \phi_T^{u_{j-1}}\|_{H^k(\mathbb{T}^3)} < \varepsilon.$$

Combining this with the above-constructed solution u_N, we get the approximate controllability of (0.5) by a control valued in $E(\mathcal{K}) = E$. The proofs of convexification and extension principles are strongly inspired by [Shi06].

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Notation

We denote by $\mathbb{T}^d$ the standard d-dimensional torus $\mathbb{R}^d/2\pi\mathbb{Z}^d$. It is endowed with the metric and the measure induced by the usual Euclidean metric and the Lebesgue measure on $\mathbb{R}^d$. More precisely, if $\Pi : \mathbb{R}^d \to \mathbb{T}^d$ denotes the canonical projection, we have

$$d(x,y) = \inf\{|\tilde{x}-\tilde{y}| : \Pi\tilde{x} = x, \Pi\tilde{y} = y, \tilde{x}, \tilde{y} \in \mathbb{R}^d\} \quad \text{for any } x, y \in \mathbb{T}^d,$$
$$\mathrm{d}(A) = (2\pi)^{-d}\mathrm{d}_{\mathbb{R}^d}(\Pi^{-1}(A) \cap [0,2\pi]^d) \quad \text{for any Borel subset } A \subset \mathbb{T}^d,$$

where $|x| = |x_1| + \cdots + |x_d|, x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $\mathrm{d}_{\mathbb{R}^d}$ is the Lebesgue measure on $\mathbb{R}^d$.

$L^p(\mathbb{T}^d, \mathbb{R}^d)$ and $H^s(\mathbb{T}^d, \mathbb{R}^d)$ stand for spaces of vector functions $u = (u_1, \ldots, u_d)$ with components in the usual Lebesgue and Sobolev spaces on $\mathbb{T}^d$.

$C^{k,\lambda}(\mathbb{T}^d, \mathbb{R}^d)$, $k \geq 0$, $\lambda \in (0,1]$ is the space of vector functions $u = (u_1, \ldots, u_d)$ with components that are continuous on $\mathbb{T}^d$ together with their derivatives up to order k, and whose derivatives of order k are Hölder-continuous of exponent λ, equipped with the norm

$$\|u\|_{C^{k,\lambda}} := \sum_{|\alpha| \leq k} \sup_{x \in \mathbb{T}^d} |D^\alpha u(x)| + \sum_{|\alpha|=k} \sup_{x,y \in \mathbb{T}^d, x \neq y} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{d(x,y)^\lambda}.$$

$H^k_\sigma(\mathbb{T}^d, \mathbb{R}^d) := H^k(\mathbb{T}^d, \mathbb{R}^d) \cap H$ and $C^{k,\lambda}_\sigma(\mathbb{T}^d, \mathbb{R}^d) := C^{k,\lambda}(\mathbb{T}^d, \mathbb{R}^d) \cap H$, where H is given by (0.4) (with d instead of 3). In what follows, when the space dimension d is 3, we shall write $L^p, H^k, \ldots$ instead of $L^p(\mathbb{T}^3, \mathbb{R}^3), H^k(\mathbb{T}^3, \mathbb{R}^3), \ldots$.

$C^1(\mathbb{T}^d)$ is the space of continuously differentiable maps from $\mathbb{T}^d$ to $\mathbb{T}^d$ endowed with the usual distance $\|\psi_1 - \psi_2\|_{C^1(\mathbb{T}^d)}$, $\psi_1, \psi_2 \in C^1(\mathbb{T}^d)$.

Let X be a Banach space endowed with a norm $\|\cdot\|_X$ and $J_T := [0,T]$. For $1 \leq p < \infty$, let $L^p(J_T, X)$ be the space of measurable functions $u : J_T \to X$ such that

$$\|u\|_{L^p(J_T,X)} := \left(\int_0^T \|u(s)\|_X^p \mathrm{d}s\right)^{\frac{1}{p}} < \infty.$$

The spaces $C(J_T, X)$ and $W^{k,p}(J_T, X)$ are defined in a similar way. We define the *relaxation* norm on $L^1(J_T, X)$ by

$$|||u|||_{T,X} := \sup_{t \in J_T} \left\| \int_0^t u(s)\mathrm{d}s \right\|_X.$$

A mapping $\psi : \mathbb{T}^d \to \mathbb{T}^d$ is volume-preserving if $\mathrm{d}(\psi^{-1}(A)) = \mathrm{d}(A)$ for any Borel subset $A \subset \mathbb{T}^d$. We denote by $\mathrm{SDiff}(\mathbb{T}^d)$ be the group of all diffeomorphisms on $\mathbb{T}^d$ preserving the orientation and volume and isotopic to the identity, i.e., $\mathrm{SDiff}(\mathbb{T}^d)$ is the set of all functions $\psi : \mathbb{T}^d \to \mathbb{T}^d$ such that there is a path $h \in W^{1,\infty}(J_1, C^1(\mathbb{T}^d))$ with $h(0,x) = x$, $h(1,x) = \psi(x)$ for all $x \in \mathbb{T}^d$, and $h(t,\cdot)$ is a diffeomorphism on $\mathbb{T}^d$ preserving the orientation and volume for all $t \in J_1$.

1 Preliminaries

1.1 Particle trajectories

In this section, we study some existence and stability properties for the Lagrangian trajectories, which are essential for the proofs of the main results. Let us fix a time $T > 0$ and an integer $d \geq 1$. For any vector field $u \in L^1(J_T, C^1(\mathbb{T}^d, \mathbb{R}^d))$, we consider the following ordinary differential equation in $\mathbb{T}^d$

$$\dot{x} = u(t, x). \tag{1.1}$$

By standard methods, one can show that for any $y \in \mathbb{T}^d$ this equation admits a unique solution $x \in W^{1,1}(J_T, \mathbb{T}^d)$ such that $x(0) = y$ (e.g., see [Hal80]). Moreover, if $\phi_t^u : \mathbb{T}^d \to \mathbb{T}^d, t \in J_T$ is the corresponding flow sending y to $x(t)$, then ϕ_t^u is a C^1-diffeomorphism on $\mathbb{T}^n$ and

$$\phi_T^{\cdot} : L^1(J_T, C^1(\mathbb{T}^d, \mathbb{R}^d)) \to C(J_T, C^1(\mathbb{T}^d)), \quad u \mapsto \phi^u \quad \text{ is continuous.} \tag{1.2}$$

We shall also use the following stability property with respect to a weaker norm (cf. Chapter 4 in [Gam78]).

Lemma 1.1. *For any $\lambda \in (0, 1]$ and $R > 0$, there is $C := C(R, \lambda, T) > 0$ such that*

$$\|\phi^u - \phi^{\hat{u}}\|_{L^\infty(J_T, C^1(\mathbb{T}^d))} \leq C \|\!| u - \hat{u} |\!\|_{T, C^1(\mathbb{T}^d, \mathbb{R}^d)}^{\lambda/2} \tag{1.3}$$

for any $u, \hat{u} \in L^\infty(J_T, C^{1,\lambda}(\mathbb{T}^d, \mathbb{R}^d))$ verifying

$$\|u\|_{L^\infty(J_T, C^{1,\lambda}(\mathbb{T}^d, \mathbb{R}^d))} + \|\hat{u}\|_{L^\infty(J_T, C^{1,\lambda}(\mathbb{T}^d, \mathbb{R}^d))} \leq R.$$

Proof. Clearly, it suffices to prove this lemma in the case when $\mathbb{T}^d$ is replaced by $\mathbb{R}^d$ and $\operatorname{supp} u(t, \cdot), \operatorname{supp} \hat{u}(t, \cdot) \subset K$ for some compact $K \subset \mathbb{R}^d$ for all $t \in J_T$.

Step 1. Let us show that there is a constant $C := C(R, T) > 0$ such that

$$\|\phi^u - \phi^{\hat{u}}\|_{L^\infty(J_T \times \mathbb{R}^d)} \leq C \|\!| u - \hat{u} |\!\|_{T, L^\infty(\mathbb{R}^d)}^{1/2}. \tag{1.4}$$

Indeed, we have

$$\begin{aligned} \|\phi_t^u - \phi_t^{\hat{u}}\|_{L^\infty(\mathbb{R}^d)} &= \left\| \int_0^t (u(s, \phi_s^u) - \hat{u}(s, \phi_s^{\hat{u}})) \mathrm{d}s \right\|_{L^\infty(\mathbb{R}^d)} \\ &\leq \left\| \int_0^t (u(s, \phi_s^u) - u(s, \phi_s^{\hat{u}})) \mathrm{d}s \right\|_{L^\infty(\mathbb{R}^d)} \\ &\quad + \left\| \int_0^t (u(s, \phi_s^{\hat{u}}) - \hat{u}(s, \phi_s^{\hat{u}})) \mathrm{d}s \right\|_{L^\infty(\mathbb{R}^d)} =: G_1 + G_2. \end{aligned} \tag{1.5}$$

Then

$$G_1 \leq \|u\|_{L^\infty(J_T, C^1(\mathbb{R}^d))} \int_0^t \|\phi_s^u - \phi_s^{\hat{u}}\|_{L^\infty(\mathbb{R}^d)} \mathrm{d}s. \tag{1.6}$$

To estimate G_2, let us first note that for any $\eta > 0$

$$\sup_{t_1,t_2\in J_T,|t_1-t_2|\le\eta} \|\phi^{\hat{u}}_{t_1} - \phi^{\hat{u}}_{t_2}\|_{L^\infty(\mathbb{R}^d)} \le \eta\|\hat{u}\|_{L^\infty(J_T\times\mathbb{R}^d)}. \tag{1.7}$$

Taking a partition $\tau_i = it/n, i = 0,\ldots,n$ and using (1.7), we get

$$\begin{aligned}
G_2 \le& \sum_{i=1}^n \left\| \int_{\tau_{i-1}}^{\tau_i} (u(s,\phi^{\hat{u}}_s) - u(s,\phi^{\hat{u}}_{\tau_{i-1}}))\mathrm{d}s \right\|_{L^\infty(\mathbb{R}^d)} \\
&+ \sum_{i=1}^n \left\| \int_{\tau_{i-1}}^{\tau_i} (\hat{u}(s,\phi^{\hat{u}}_s) - \hat{u}(s,\phi^{\hat{u}}_{\tau_{i-1}}))\mathrm{d}s \right\|_{L^\infty(\mathbb{R}^d)} \\
&+ \sum_{i=1}^n \left\| \int_{\tau_{i-1}}^{\tau_i} (u(s,\phi^{\hat{u}}_{\tau_{i-1}}) - \hat{u}(s,\phi^{\hat{u}}_{\tau_{i-1}}))\mathrm{d}s \right\|_{L^\infty(\mathbb{R}^d)} \\
\le& \frac{T^2}{n}\|\hat{u}\|_{L^\infty(J_T\times\mathbb{R}^d)}(\|u\|_{L^\infty(J_T,C^1(\mathbb{R}^d))} + \|\hat{u}\|_{L^\infty(J_T,C^1(\mathbb{R}^d))}) \\
&+ 2n|\!|\!|u-\hat{u}|\!|\!|_{T,L^\infty(\mathbb{R}^d)}.
\end{aligned} \tag{1.8}$$

If $|\!|\!|u-\hat{u}|\!|\!|_{T,L^\infty(\mathbb{R}^d)} = 0$, then (1.3) holds trivially. Assume that $|\!|\!|u-\hat{u}|\!|\!|_{T,L^\infty(\mathbb{R}^d)} > 0$. Choosing[1] $n := [|\!|\!|u-\hat{u}|\!|\!|^{-1/2}_{T,L^\infty(\mathbb{R}^d)}]$, we see that

$$G_2 \le C|\!|\!|u-\hat{u}|\!|\!|^{1/2}_{T,L^\infty(\mathbb{R}^d)}.$$

Combining this with (1.5) and (1.6) and applying the Gronwall inequality, we obtain (1.4).

Step 2. For $j = 1,\ldots,d$, we have

$$\begin{aligned}
\|\partial_j\phi^u_t - \partial_j\phi^{\hat{u}}_t\|_{L^\infty(\mathbb{R}^d)} =& \left\| \int_0^t (\langle\nabla u(s,\phi^u_s),\partial_j\phi^u_s\rangle - \langle\nabla\hat{u}(s,\phi^{\hat{u}}_s),\partial_j\phi^{\hat{u}}_s\rangle)\mathrm{d}s \right\|_{L^\infty(\mathbb{R}^d)} \\
\le& \left\| \int_0^t (\langle\nabla u(s,\phi^u_s),\partial_j\phi^u_s - \partial_j\phi^{\hat{u}}_s\rangle)\mathrm{d}s \right\|_{L^\infty(\mathbb{R}^d)} \\
&+ \left\| \int_0^t (\langle\nabla u(s,\phi^u_s) - \nabla u(s,\phi^{\hat{u}}_s),\partial_j\phi^{\hat{u}}_s\rangle)\mathrm{d}s \right\|_{L^\infty(\mathbb{R}^d)} \\
&+ \left\| \int_0^t (\langle\nabla u(s,\phi^{\hat{u}}_s) - \nabla\hat{u}(s,\phi^{\hat{u}}_s),\partial_j\phi^{\hat{u}}_s\rangle)\mathrm{d}s \right\|_{L^\infty(\mathbb{R}^d)} \\
=:& I_1 + I_2 + I_3.
\end{aligned} \tag{1.9}$$

It is easy to verify that there is a constant $C_1 := C_1(R,T) > 0$ such that

$$\|\phi^{\hat{u}}\|_{L^\infty(J_T,C^1(\mathbb{R}^d))} + \|\dot{\phi}^{\hat{u}}\|_{L^\infty(J_T,C^1(\mathbb{R}^d))} \le C_1. \tag{1.10}$$

[1] Here $[a]$ stands for the integer part of $a \in \mathbb{R}$.

Using this and (1.4), we get that

$$I_1 \leq R \int_0^t \|\partial_j \phi_s^u - \partial_j \phi_s^{\hat{u}}\|_{L^\infty(\mathbb{R}^d)} \mathrm{d}s,$$

$$I_2 \leq C_1 R \int_0^t \|\phi_s^u - \phi_s^{\hat{u}}\|_{L^\infty(\mathbb{R}^d)}^\lambda \mathrm{d}s \leq C_2 \|\!|u - \hat{u}|\!\|_{T, L^\infty(\mathbb{R}^d)}^{\lambda/2}.$$

To estimate I_3, we integrate by parts and use (1.10)

$$\begin{aligned} I_3 &\leq \left\| \langle \int_0^t (\nabla u(s, \phi_s^{\hat{u}}) - \nabla \hat{u}(s, \phi_s^{\hat{u}})) \mathrm{d}s, \partial_j \phi_t^{\hat{u}} \rangle \right\|_{L^\infty(\mathbb{R}^d)} \\ &\quad + \left\| \int_0^t \left(\langle \int_0^s (\nabla u(\theta, \phi_\theta^{\hat{u}}) - \nabla \hat{u}(\theta, \phi_\theta^{\hat{u}})) \mathrm{d}\theta, \partial_j \dot{\phi}_s^{\hat{u}} \rangle \right) \mathrm{d}s \right\|_{L^\infty(\mathbb{R}^d)} \\ &\leq C_3 \sup_{s \in [0,t]} \left\| \int_0^s (\nabla u(\theta, \phi_\theta^{\hat{u}}) - \nabla \hat{u}(\theta, \phi_\theta^{\hat{u}})) \mathrm{d}\theta \right\|_{L^\infty(\mathbb{R}^d)}. \end{aligned}$$

Repeating the arguments used in (1.8) and using the fact that ∇u and $\nabla \hat{u}$ are Hölder continuous with exponent λ, we obtain that

$$\begin{aligned} \sup_{s \in [0,t]} \left\| \int_0^s (\nabla u(\theta, \phi_\theta^{\hat{u}}) - \nabla \hat{u}(\theta, \phi_\theta^{\hat{u}})) \mathrm{d}\theta \right\|_{L^\infty(\mathbb{R}^d)} &\leq \frac{C_4}{n^\lambda} + 2n \|\!|u - \hat{u}|\!\|_{T, C^1(\mathbb{R}^d)} \\ &\leq C_5 \|\!|u - \hat{u}|\!\|_{T, C^1(\mathbb{R}^d)}^{\lambda/2} \end{aligned}$$

for $n := [\|\!|u - \hat{u}|\!\|_{T, C^1(\mathbb{R}^d)}^{-1/2}]$. Combining this with the estimates for I_1, I_2 and (1.9), and applying the Gronwall inequality, we arrive at the required result. □

By the Liouville theorem, if we assume additionally that u is divergence-free, then the flow ϕ_t^u preserves the orientation and the volume. Thus if $u \in L^\infty(J_T, C_\sigma^1(\mathbb{T}^d, \mathbb{R}^d))$, then $\phi_t^u \in \mathrm{SDiff}(\mathbb{T}^d)$ for any $t \in J_T$. The following proposition shows that, using a suitable divergence-free field u, the flow ϕ_t^u can be driven approximately to any position $\psi \in \mathrm{SDiff}(\mathbb{T}^d)$ at time T.

Proposition 1.2. *For any $\varepsilon > 0, k > 1 + d/2$, $u_0, u_1 \in H_\sigma^k(\mathbb{T}^d, \mathbb{R}^d)$, and $\psi \in \mathrm{SDiff}(\mathbb{T}^d)$, there is a vector field $u \in C^\infty(J_T, H_\sigma^k(\mathbb{T}^d, \mathbb{R}^d))$ such that $u(0) = u_0, u(T) = u_1$, and*

$$\|\phi_T^u - \psi\|_{C^1(\mathbb{T}^d)} < \varepsilon.$$

Proof. Step 1. We first forget about the endpoint conditions $u(0) = u_0, u(T) = u_1$ and show that there is a divergence-free vector field $\hat{u} \in C^\infty(\mathbb{R} \times \mathbb{T}^d, \mathbb{R}^d)$ such that

$$\|\phi_T^{\hat{u}} - \psi\|_{C^1(\mathbb{T}^d)} < \varepsilon/2.$$

Since $\psi \in \mathrm{SDiff}(\mathbb{T}^d)$, there is a path $h \in W^{1,\infty}(J_T, C^1(\mathbb{T}^d))$ such that $h(0, x) = x$, $h(T, x) = \psi(x)$ for all $x \in \mathbb{T}^d$, and $h(t, \cdot)$ is a C^1-diffeomorphism on $\mathbb{T}^d$ preserving the orientation and the volume for all $t \in J_T$. Let us define the vector field $\hat{u}(t, x) = \dot{h}(t, h^{-1}(t, x))$. Then we have $\hat{u} \in L^\infty(J_T, C^1(\mathbb{T}^d, \mathbb{R}^d))$ and

$h(t,x)=\phi_t^{\hat u}(x), t\in J_T$. As $\phi_t^{\hat u}$ preserves the orientation and the volume, for any $g\in C^1(\mathbb{T}^d,\mathbb{R})$

$$
\begin{aligned}
0=\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d} g(\phi_t^{\hat u}(y))\mathrm{d}y &= \int_{\mathbb{T}^d}\langle\nabla g(\phi_t^{\hat u}(y)),\dot\phi_t^{\hat u}(y)\rangle\mathrm{d}y\\
&=\int_{\mathbb{T}^d}\langle\nabla g(\phi_t^{\hat u}(y)),\hat u(t,\phi_t^{\hat u}(y))\rangle\mathrm{d}y=\int_{\mathbb{T}^d} g(y)\,\mathrm{div}\,\hat u(t,y)\mathrm{d}y.
\end{aligned}
$$

This shows that $\hat u$ is divergence-free. Taking a sequence of mollifying kernels $\rho_n, n\geq 1$, we consider $\hat u_n:=\rho_n*\hat u\in C^\infty(\mathbb{R}\times\mathbb{T}^d,\mathbb{R}^d)$. Then $\hat u_n$ is also divergence-free and $\|\hat u_n-\hat u\|_{L^\infty(J_T,C^1(\mathbb{T}^d,\mathbb{R}^d))}\to 0$ as $n\to\infty$. By (1.2), this implies that $\|\phi_T^{\hat u_n}-\phi_T^{\hat u}\|_{C^1(\mathbb{T}^d)}\to 0$ as $n\to\infty$. Since $\phi_T^{\hat u}=\psi$, we get the required result with $\hat u=\hat u_n$ for sufficiently large $n\geq 1$.

Step 2. By the Sobolev embedding, $H^k\subset C^1(\mathbb{T}^d), k>1+d/2$ (e.g., see [Ada75]). For any $\delta>0$, we take an arbitrary $u\in C^\infty(J_T,H^k_\sigma(\mathbb{T}^d,\mathbb{R}^d))$ satisfying

$$
\begin{aligned}
&u(0)=u_0,\quad u(T)=u_1,\\
&\|u-\hat u\|_{L^1(J_T,C^1(\mathbb{T}^d,\mathbb{R}^d))}<\delta.
\end{aligned}
$$

Then by Step 1 and (1.2), we have

$$
\begin{aligned}
\|\phi_T^u-\psi\|_{C^1(\mathbb{T}^d)} &\leq \|\phi_T^u-\phi_T^{\hat u}\|_{C^1(\mathbb{T}^d)}+\|\phi_T^{\hat u}-\psi\|_{C^1(\mathbb{T}^d)}\\
&<\varepsilon/2+\varepsilon/2=\varepsilon
\end{aligned}
$$

for sufficiently small $\delta>0$.

□

1.2 Existence of strong solutions

In what follows, we shall assume that $d=3, k\geq 3$, and $\nu=1$. In this section, we prove a perturbative result on existence of strong solutions for the evolution equation

$$\dot u+Lu+B(u)=g, \tag{1.11}$$

where $B(a,b)=\Pi\{\langle a,\nabla\rangle b\}$ and $B(a)=B(a,a)$. Along with (1.11), we consider the following more general equation

$$\dot u+L(u+\zeta)+B(u+\zeta)=g. \tag{1.12}$$

Let us fix any $T>0$ and introduce the space $\mathcal{X}_{T,k}:=C(J_T,H^k_\sigma)\cap L^2(J_T,H^{k+1}_\sigma)$ endowed with the norm

$$\|u\|_{\mathcal{X}_{T,k}}:=\|u\|_{L^\infty(J_T,H^k)}+\|u\|_{L^2(J_T,H^{k+1})}.$$

The following result is a version of Theorem 1.8 and Remark 1.9 in [Shi06] and Theorem 2.1 in [Ner10] in the case of the 3D NS system in the spaces $H^k, k\geq 3$. For the sake of completeness, we give all the details of the proof, even though it is very close to the proofs of the previous results.

Theorem 1.3. *Suppose that for some functions $\hat{u}_0 \in H^k_\sigma$, $\hat{\zeta} \in L^4(J_T, H^{k+1}_\sigma)$, and $\hat{g} \in L^2(J_T, H^{k-1}_\sigma)$ problem (1.12), (0.2) with $u_0 = \hat{u}_0$, $\zeta = \hat{\zeta}$, and $g = \hat{g}$ has a solution $\hat{u} \in \mathcal{X}_{T,k}$. Then there are positive constants δ and C depending only on*

$$\|\hat{\zeta}\|_{L^4(J_T,H^{k+1})} + \|\hat{g}\|_{L^2(J_T,H^{k-1})} + \|\hat{u}\|_{\mathcal{X}_{T,k}}$$

such that the following statements hold.

(i) If $u_0 \in H^k_\sigma$, $\zeta \in L^4(J_T, H^{k+1}_\sigma)$, and $g \in L^2(J_T, H^{k-1}_\sigma)$ satisfy the inequality

$$\|u_0 - \hat{u}_0\|_k + \|\zeta - \hat{\zeta}\|_{L^4(J_T,H^{k+1})} + \|g - \hat{g}\|_{L^2(J_T,H^{k-1})} < \delta, \tag{1.13}$$

then problem (1.12), (0.2) has a unique solution $u \in \mathcal{X}_{T,k}$.

(ii) Let

$$\mathcal{R} : H^k_\sigma \times L^4(J_T, H^{k+1}_\sigma) \times L^2(J_T, H^{k-1}_\sigma) \to \mathcal{X}_{T,k}$$

be the operator that takes each triple (u_0, ζ, g) satisfying (1.13) to the solution u of (1.12), (0.2). Then

$$\begin{aligned}\|\mathcal{R}(u_0, \zeta, g) - \mathcal{R}(\hat{u}_0, \hat{\zeta}, \hat{g})\|_{\mathcal{X}_{T,k}} \le C\big(&\|u_0 - \hat{u}_0\|_k \\ &+ \|\zeta - \hat{\zeta}\|_{L^4(J_T,H^{k+1})} + \|g - \hat{g}\|_{L^2(J_T,H^{k-1})}\big).\end{aligned}$$

Proof. We use the following standard estimates for the bilinear form B

$$\|B(a, b)\|_k \le C\|a\|_k\|b\|_{k+1} \qquad \text{for } k \ge 2, \tag{1.14}$$

$$|(B(a, b), L^k b)| \le C\|a\|_k\|b\|^2_k \qquad \text{for } k \ge 3 \tag{1.15}$$

for any $a \in H^k_\sigma$ and $b \in H^{k+1}_\sigma$ (see Chapter 6 in [CF88]). We are looking for a solution of (1.12), (0.2) of the form $u = \hat{u} + w$. We have the following equation for w:

$$\begin{aligned}&\dot{w} + L(w + \eta) + B(w + \eta, \hat{u} + \hat{\zeta}) + B(\hat{u} + \hat{\zeta}, w + \eta) + B(w + \eta) = q,\\ &w(0, x) = w_0(x),\end{aligned} \tag{1.16}$$

where $w_0 = u_0 - \hat{u}_0$, $\eta = \zeta - \hat{\zeta}$, and $q = g - \hat{g}$. Setting $\tilde{B}(u, v) = B(u, v) + B(v, u)$, we get that

$$\dot{w}+Lw+B(w)+\tilde{B}(w,\eta)+\tilde{B}(w,\hat{u})+\tilde{B}(w,\hat{\zeta})=q-(L\eta+B(\eta)+\tilde{B}(\hat{u},\eta)+\tilde{B}(\hat{\zeta},\eta)), \tag{1.17}$$

Using (1.14), we see that for any $\varepsilon > 0$, we can choose $\delta > 0$ in (1.13) such that

$$\|w_0\|_k + \|q - (L\eta + B(\eta) + \tilde{B}(\hat{u}, \eta) + \tilde{B}(\hat{\zeta}, \eta))\|_{L^2(J_T,H^{k-1})} < \varepsilon.$$

Then a standard existence result implies that system (1.17), (1.16) has a solution $w \in \mathcal{X}_{T,k}$ (see [Tay97]).

To prove (ii), we multiply (1.17) by $L^k w$ and use estimates (1.14) and (1.15)

$$\frac{1}{2}\frac{d}{dt}\|w\|_k^2+\|w\|_{k+1}^2 \le C\Big(\|w\|_k^3+\|w\|_{k+1}\|w\|_k\big(\|\eta\|_k+\|\hat{u}\|_k+\|\hat{\zeta}\|_k\big)$$
$$+\|w\|_{k+1}\big(\|q\|_{k-1}+\|\eta\|_{k+1}+\|\eta\|_k(\|\eta\|_{k-1}+\|\hat{u}\|_k+\|\hat{\zeta}\|_k)\big)\Big).$$

This implies that

$$\frac{1}{2}\frac{d}{dt}\|w\|_k^2+\frac{1}{2}\|w\|_{k+1}^2 \le C_1\Big(\|w\|_k^3+\|w\|_k^2\big(\|\eta\|_k^2+\|\hat{u}\|_k^2+\|\hat{\zeta}\|_k^2\big)$$
$$+\Big[\|q\|_{k-1}^2+\|\eta\|_{k+1}^4+\|\eta\|_k^2(\|\hat{u}\|_k^2+\|\hat{\zeta}\|_k^2)\Big]\Big).$$

Integrating this inequality and setting

$$A:=\|w_0\|_k^2+\int_0^T\Big[\|q\|_{k-1}^2+\|\eta\|_{k+1}^4+\|\eta\|_k^2(\|\hat{u}\|_k^2+\|\hat{\zeta}\|_k^2)\Big]\mathrm{d}t,$$

we obtain

$$\|w\|_k^2+\int_0^t\|w\|_{k+1}^2 \le 2A+2C_1\int_0^t\Big(\|w\|_k^3+\|w\|_k^2\big(\|\eta\|_k^2+\|\hat{u}\|_k^2+\|\hat{\zeta}\|_k^2\big)\Big)\mathrm{d}t. \tag{1.18}$$

The Gronwall inequality gives that

$$\|w\|_k^2 \le C_2 A + C_2\int_0^t \|w(s,\cdot)\|_k^3\mathrm{d}s,$$

where C_2 depends only on $\|\hat{u}\|_{L^2(J_T,H^k)}+\|\hat{\zeta}\|_{L^2(J_T,H^k)}$. Another application of the Gronwall inequality implies that

$$\|w(t)\|_k \le \frac{C_2A}{1-C_2^2At} \le 2C_2A \ \text{ for any } \ t\le \frac{1}{2C_2^2A}. \tag{1.19}$$

Let us choose $\delta>0$ so small that $\frac{1}{2C_2^2A}\ge T$. Using (1.19) and (1.18), and choosing $\delta>0$ sufficiently small, we get for any $t\in J_T$

$$\|w\|_k^2+\int_0^t\|w\|_{k+1}^2 \le C_3A \le C_4\big(\|w_0\|_k^2+\|\eta\|_{L^4(J_T,H^{k+1})}^2+\|q\|_{L^2(J_T,H^{k-1})}^2\big).$$

This completes the proof of the theorem. □

2 Main results

2.1 Approximate controllability of the NS system

In this section, we state the main results of this paper. Let us fix any $T>0$ and $k\ge 3$, and consider the NS system

$$\dot{u}+Lu+B(u)=h(t)+\eta(t), \tag{2.1}$$
$$u(0,x)=u_0(x), \tag{2.2}$$

where $h \in L^2(J_T, H^{k-1}_\sigma)$ and $u_0 \in H^k_\sigma$ are given functions and η is a control taking values in a finite-dimensional space $E \subset H^{k+1}_\sigma$. We denote by $\Theta(h, u_0)$ the set of functions $\eta \in L^2(J_T, H^{k-1}_\sigma)$ for which (2.1), (2.2) has a solution u in $\mathcal{X}_{T,k}$. By Theorem 1.3, $\Theta(h, u_0)$ is an open subset of $L^2(J_T, H^{k-1}_\sigma)$. Recall that $\mathcal{R}(\cdot,\cdot,\cdot)$ is the operator defined in Theorem 1.3. To simplify notation, we write $\mathcal{R}(\cdot,\cdot)$ instead of $\mathcal{R}(\cdot, 0, \cdot)$. The embedding $H^3 \subset C^{1,1/2}$ implies that the flow $\phi_t^{\mathcal{R}(u_0,h+\eta)}$ is well defined for any $\eta \in \Theta(h, u_0)$ and $t \in J_T$. We set

$$Y_{T,k} := X_{T,k} \cap W^{1,2}(J_T, H^{k-1}_\sigma).$$

We shall use the following notion of controllability.

Definition 2.1. Equation (2.1) is said to be *controllable* at time T by an E-valued control if for any $\varepsilon > 0$ and any $\varphi \in Y_{T,k}$ there is a control $\eta \in \Theta(h, u_0) \cap L^2(J_T, E)$ such that

$$\|\mathcal{R}_T(u_0, h+\eta) - \varphi(T)\|_k + \|\!|\mathcal{R}(u_0, h+\eta) - \varphi|\!\|_{T,k} + \|\phi^{\mathcal{R}(u_0,h+\eta)} - \phi^\varphi\|_{L^\infty(J_T,C^1)} < \varepsilon,$$

where $u_0 = \varphi(0)$ and $|\!\|\cdot|\!\|_{T,k} := |\!\|\cdot|\!\|_{T,H^k}$.

Let us recall some notation introduced in [AS05, AS06], and [Shi06]. For any finite-dimensional subspace $E \subset H^{k+1}_\sigma$, we denote by $\mathcal{F}(E)$ the largest vector space $F \subset H^{k+1}_\sigma$ such that for any $\eta_1 \in F$ there are vectors[2] $\eta, \zeta^1, \dots, \zeta^n \in E$ satisfying the relation

$$\eta_1 = \eta - \sum_{i=1}^n B(\zeta^i). \tag{2.3}$$

As E is a finite-dimensional subspace and B is a bilinear operator, the set of all vectors $\eta_1 \in H^{k+1}_\sigma$ of the form (2.3) is contained in a finite-dimensional space. It is easy to see that if subspaces $G_1, G_2 \subset H^{k+1}_\sigma$ are composed of elements η_1 of the form (2.3), then so does $G_1 + G_2$. Thus the space $\mathcal{F}(E)$ is well defined. We define E_j by the rule

$$E_0 = E, \quad E_j = \mathcal{F}(E_{j-1}) \quad \text{for} \quad j \geq 1, \quad E_\infty = \bigcup_{j=1}^\infty E_j. \tag{2.4}$$

Clearly, E_j is a non-decreasing sequence of subspaces. We say that E is *saturating* in H^{k-1}_σ if E_∞ is dense in H^{k-1}_σ. The following theorem is the main result of this paper.

Theorem 2.2. *Assume that E is a finite-dimensional subspace of H^{k+1}_σ and $h \in L^2(J_T, H^{k-1}_\sigma)$. If E is saturating in H^{k-1}_σ, then (2.1) with $\eta \in C^\infty(J_T, E)$ is controllable at time T.*

We have the following two corollaries of this result.

[2] The integer n may depend on η_1.

Corollary 2.3. *Under the conditions of Theorem 2.2, if E is saturating in H_σ^{k-1}, then for any $\varepsilon > 0$, $u_0, u_1 \in H_\sigma^k$, and $\psi \in \mathrm{SDiff}(\mathbb{T}^3)$ there is a control $\eta \in \Theta(h, u_0) \cap C^\infty(J_T, E)$ such that*

$$\|\mathcal{R}_T(u_0, h+\eta) - u_1\|_k + \|\phi_T^{\mathcal{R}(u_0,h+\eta)} - \psi\|_{C^1} < \varepsilon.$$

Let us denote by $\mathrm{VPM}(\mathbb{T}^3)$ the set of all volume-preserving mappings from $\mathbb{T}^3$ to $\mathbb{T}^3$. According to Corollary 1.5 in [BG03] and Theorem 2.1 in [Shn85], we have that $\mathrm{VPM}(\mathbb{T}^3)$ is the closure of $\mathrm{SDiff}(\mathbb{T}^3)$ in $L^p(\mathbb{T}^3)$ for any $p \in [1, +\infty)$. Thus we get the following result.

Corollary 2.4. *Under the conditions of Theorem 2.2, if E is saturating in H_σ^{k-1}, then for any $\varepsilon > 0$, $p \in [1, +\infty)$, $u_0, u_1 \in H_\sigma^k$, and $\psi \in \mathrm{VPM}(\mathbb{T}^3)$ there is a control $\eta \in \Theta(h, u_0) \cap C^\infty(J_T, E)$ such that*

$$\|\mathcal{R}_T(u_0, h+\eta) - u_1\|_k + \|\phi_T^{\mathcal{R}(u_0,h+\eta)} - \psi\|_{L^p} < \varepsilon.$$

The rest of this subsection is devoted to the proofs of Theorem 2.2 and Corollary 2.3. They are based on the following result which is proved in Section 3.

Theorem 2.5. *Under the conditions of Theorem 2.2, for any $\varepsilon > 0$, $u_0 \in H_\sigma^k$, and $\eta_1 \in \Theta(h, u_0) \cap L^2(J_T, E_1)$ there is $\eta \in \Theta(h, u_0) \cap C^\infty(J_T, E)$ such that*

$$\|\mathcal{R}_T(u_0, h+\eta_1) - \mathcal{R}_T(u_0, h+\eta)\|_k + |||\mathcal{R}(u_0, h+\eta_1) - \mathcal{R}(u_0, h+\eta)|||_{T,k} + \|\phi^{\mathcal{R}(u_0,h+\eta_1)} - \phi^{\mathcal{R}(u_0,h+\eta)}\|_{L^\infty(J_T, C^1)} < \varepsilon.$$

Proof of Theorem 2.2. Let us take any $\varepsilon > 0, \delta > 0$, and $\varphi \in Y_{T,k}$. Then

$$\eta_0 := \dot{\varphi} + L\varphi + B(\varphi) - h$$

belongs to $\Theta(u_0, h)$ and $\varphi(t) = \mathcal{R}_t(u_0, h+\eta_0)$ for any $t \in J_T$, where $u_0 = \varphi(0)$. Since E_∞ is dense in H_σ^{k-1}, we have that

$$\|P_{E_N}\eta_0 - \eta_0\|_{L^2(J_T, H^{k-1})} \to 0 \text{ as } N \to \infty.$$

By Theorem 1.3, for sufficiently large N, we have $P_{E_N}\eta_0 \in \Theta(h, u_0)$ and

$$\|\mathcal{R}(u_0, h + P_{E_N}\eta_0) - \varphi\|_{X_{T,k}} < \delta.$$

By (1.2), we can choose $\delta > 0$ so small that

$$\|\phi^{\mathcal{R}(u_0, h+P_{E_N}\eta_0)} - \phi^\varphi\|_{L^\infty(J_T, C^1)} < \varepsilon.$$

Applying N times Theorem 2.5, we complete the proof of Theorem 2.2.

□

Proof of Corollary 2.3. Let us take any $\varepsilon > 0$, $\psi \in \mathrm{SDiff}(\mathbb{T}^3)$, and $u_0, u_1 \in H_\sigma^k$. By Proposition 1.2, there is a vector field $u \in C^\infty(J_T, H_\sigma^k)$ such that $u(0) = u_0, u(T) = u_1$, and

$$\|\phi_T^u - \psi\|_{C^1} < \varepsilon. \tag{2.5}$$

Applying Theorem 2.2 with $\varphi = u$, we find a control $\eta \in \Theta(h, u_0) \cap C^\infty(J_T, E)$ such that

$$\|\mathcal{R}_T(u_0, h+\eta) - u(T)\|_k + \|\phi_T^{\mathcal{R}(u_0,h+\eta)} - \phi_T^u\|_{C^1} < \varepsilon.$$

Combining this with (2.5), we get the required result.

□

2.2 Examples of saturating spaces

In this section, we provide three types of examples of saturating spaces which ensure the controllability of the 3D NS system in the sense of Definition 2.1.

2.2.1 Saturating spaces associated with the generators of $\mathbb{Z}^3$

Let us first introduce some notation. For any $\ell \in \mathbb{Z}^3_*$, let us define the functions

$$c_\ell(x) = l(\ell)\cos\langle \ell, x\rangle,\ s_\ell(x) = l(\ell)\sin\langle \ell, x\rangle, \tag{2.6}$$

where $\{l(\ell), l(-\ell)\}$ is an arbitrary orthonormal basis in

$$\ell^\perp := \{x \in \mathbb{R}^3 : \langle x, \ell\rangle = 0\}.$$

Then c_ℓ and s_ℓ are eigenfunctions of L and the family $\{c_\ell, s_\ell\}_{\ell\in\mathbb{Z}^3_*}$ is an orthonormal basis in H. Let $c_0 = s_0 = 0$. For any subset $\mathcal{K} \subset \mathbb{Z}^3$, we denote

$$E(\mathcal{K}) := \operatorname{span}\{c_\ell, c_{-\ell}, s_\ell, s_{-\ell} : \ell \in \mathcal{K}\}. \tag{2.7}$$

When $\mathcal{K}$ is finite, the spaces $E_j(\mathcal{K})$ and $E_\infty(\mathcal{K})$ are defined by (2.4) with $E = E(\mathcal{K})$. We denote by $\mathbb{Z}^3_{\mathcal{K}}$ the set of all vectors $a \in \mathbb{Z}^3$ which can be represented as finite linear combination of elements of $\mathcal{K}$ with integer coefficients. We shall say that $\mathcal{K} \subset \mathbb{Z}^3$ is a *generator* if $\mathbb{Z}^3_{\mathcal{K}} = \mathbb{Z}^3$. The following theorem provides a characterisation of saturating spaces of the form (2.7).

Theorem 2.6. *For any finite set $\mathcal{K} \subset \mathbb{Z}^3$, we have the equality*

$$E(\mathbb{Z}^3_{\mathcal{K}}) = E_\infty(\mathcal{K}). \tag{2.8}$$

Moreover, $E(\mathcal{K})$ is saturating in H if and only if $\mathcal{K}$ is a generator of $\mathbb{Z}^3$. If $E(\mathcal{K})$ is saturating in H, then it is saturating in H^k_σ for any $k \geq 0$.

In [Rom04] a similar result is conjectured in the case of finite-dimensional approximations of the 3D NS system and a proof is given for the saturating property of $E(\mathcal{K})$ when $\mathcal{K} = \{(1,0,0), (0,1,0), (0,0,1)\}$[3]. A 2D version of Theorem 2.6 is established in [HM06]. In that case, the set $\mathcal{K}$ is a generator of $\mathbb{Z}^2$ containing at least two vectors with different Euclidian norms (the reader is referred to the original paper for the exact statement). The proof in the 3D case, as well as the statement of the result, differ essentially from the 2D case.

In view of Theorem 2.6, the following simple criterion is useful for constructing saturating spaces (see Section 3.7 in [Jac85]).

[3] In that case one has $\dim E(\mathcal{K}) = 12$. In Proposition 2.14, we give an example of a 6-dimensional saturating space.

Theorem 2.7. *A set $\mathcal{K} \subset \mathbb{Z}^3$ is a generator if and only if the greatest common divisor of the set $\{\det(a,b,c) : a,b,c \in \mathcal{K}\}$ is 1, where $\det(a,b,c)$ is the determinant of the matrix with rows a,b and c.*

The proof of Theorem 2.6 is deduced from the following auxiliary result.

Proposition 2.8. *Assume that $\mathcal{W} \subset \mathbb{Z}^3$ is a finite set containing a linearly independent family $\{p,q,r\} \subset \mathbb{Z}^3$. Then for any non-parallel vectors $m,n \in \mathcal{W}$ we have $\mathcal{A}_{m\pm n}, \mathcal{B}_{m\pm n} \subset E_5(\mathcal{W})$, where*

$$\mathcal{A}_\ell := \mathrm{span}\{c_\ell, c_{-\ell}\}, \quad \mathcal{B}_\ell := \mathrm{span}\{s_\ell, s_{-\ell}\}, \quad \ell \in \mathbb{Z}^3_*.$$

Proof of Proposition 2.8. We shall confine ourselves to the proof of the inclusion $\mathcal{A}_{m+n} \subset E_5(\mathcal{W})$. The other conclusions of the proposition are checked in the same way.

Step 1. We shall write $m \nparallel n$ when the vectors $m,n \in \mathbb{R}^3$ are non-parallel. For any $m,n \in \mathcal{W}$ such that $m \nparallel n$, let us denote by $d := d(m,n)$ one of the two unit vectors belonging to $m^\perp \cap n^\perp$. Let us show that

$$d\cos\langle m \pm n, x\rangle, d\sin\langle m \pm n, x\rangle \in E_1(\mathcal{W}). \tag{2.9}$$

Indeed, for any $a \in \mathbb{R}^3_*$, let us denote by P_a the orthogonal projection in $\mathbb{R}^3$ onto $a^\perp$. Then we have

$$\Pi(a\cos\langle l,x\rangle) = (P_l a)\cos\langle l,x\rangle, \quad \Pi(a\sin\langle l,x\rangle) = (P_l a)\sin\langle l,x\rangle$$

for any $l \in \mathbb{Z}^3_*$. Combining this with some trigonometric identities and the definition of B, one gets that

$$\begin{aligned} 2B(a\cos\langle m,x\rangle + b\sin\langle n,x\rangle) = {} & \cos\langle m-n,x\rangle P_{m-n}\left(\langle a,n\rangle b - \langle b,m\rangle a\right) \\ & + \cos\langle m+n,x\rangle P_{m+n}\left(\langle a,n\rangle b + \langle b,m\rangle a\right), \end{aligned} \tag{2.10}$$

for any $a \in m^\perp$ and $b \in n^\perp$ (see Step 1 of the proof of Proposition 2.8 in [Shi06]). This implies that

$$\begin{aligned} 2B(b\cos\langle n,x\rangle + a\sin\langle m,x\rangle) = {} & -\cos\langle m-n,x\rangle P_{m-n}\left(\langle a,n\rangle b - \langle b,m\rangle a\right) \\ & + \cos\langle m+n,x\rangle P_{m+n}\left(\langle a,n\rangle b + \langle b,m\rangle a\right). \end{aligned} \tag{2.11}$$

Taking the sum of (2.10) and (2.11), we obtain that

$$\begin{aligned} \cos\langle m+n,x\rangle P_{m+n}\left(\langle a,n\rangle b + \langle b,m\rangle a\right) = {} & B(a\cos\langle m,x\rangle + b\sin\langle n,x\rangle) \\ & + B(b\cos\langle n,x\rangle + a\sin\langle m,x\rangle). \end{aligned} \tag{2.12}$$

Let us fix any $\lambda \in \mathbb{R}$ and choose in this equality $a = d$ and $\langle b,m\rangle = \lambda$. This choice is possible since $m \nparallel n$. Then we have

$$\lambda d\cos\langle m+n,x\rangle = B(d\cos\langle m,x\rangle + b\sin\langle n,x\rangle) + B(b\cos\langle m,x\rangle + d\sin\langle n,x\rangle).$$

Since $\lambda \in \mathbb{R}$ is arbitrary, from the definition of $E_1(\mathcal{W})$ we get that $d\cos\langle m + n, x\rangle \in E_1(\mathcal{W})$. To prove that $d\cos\langle m - n, x\rangle \in E_1(\mathcal{W})$, it suffices to replace b by $-b$ in (2.11), take the sum of the resulting equality with (2.10):

$$\begin{aligned}\cos\langle m-n,x\rangle P_{m-n}\left(\langle a,n\rangle b - \langle b,m\rangle a\right) = B(a\cos\langle m,x\rangle + b\sin\langle n,x\rangle)\\ + B(-b\cos\langle n,x\rangle + a\sin\langle m,x\rangle),\end{aligned} \tag{2.13}$$

and choose $a = d$ and $\langle b, m\rangle = -\lambda$

$$\lambda d\cos\langle m-n,x\rangle = B(d\cos\langle m,x\rangle + b\sin\langle n,x\rangle) + B(-b\cos\langle n,x\rangle + d\sin\langle n,x\rangle).$$

The fact that $d\sin\langle m \pm n, x\rangle \in E_1(\mathcal{W})$ is proved in a similar way using the following identities

$$\begin{aligned}2B(a\cos\langle m,x\rangle + b\cos\langle n,x\rangle) &= \sin\langle m-n,x\rangle P_{m-n}\left(\langle a,n\rangle b - \langle b,m\rangle a\right)\\ &\quad - \sin\langle m+n,x\rangle P_{m+n}\left(\langle a,n\rangle b + \langle b,m\rangle a\right),\\ 2B(a\sin\langle m,x\rangle + b\sin\langle n,x\rangle) &= \sin\langle m-n,x\rangle P_{m-n}\left(\langle a,n\rangle b - \langle b,m\rangle a\right)\\ &\quad + \sin\langle m+n,x\rangle P_{m+n}\left(\langle a,n\rangle b + \langle b,m\rangle a\right).\end{aligned}$$

Step 2. Let us take any vector $r \in \mathcal{W}$ such that $\mathcal{E} := \{m, n, r\}$ is a basis in $\mathbb{R}^3$ ($\mathcal{E}$ is not necessarily a generator of $\mathbb{Z}^3$). This choice is possible, by the conditions of the proposition. For any $\alpha, \beta, \gamma \in \mathbb{R}$, we shall write $(\alpha, j, k)_{\mathcal{E}}$ instead of $\alpha m + \beta n + \gamma r$. Since

$$(1,1,0)_{\mathcal{E}} = (1,0,0)_{\mathcal{E}} + (0,1,0)_{\mathcal{E}} = m + n,$$

we get from Step 1 that

$$d(m,n)\cos\langle(1,1,0)_{\mathcal{E}}, x\rangle \in E_1(\mathcal{W}).$$

Applying (2.12), we obtain for any $b \in (0,0,1)_{\mathcal{E}}^{\perp}$

$$\begin{aligned}&\cos\langle(1,1,1)_{\mathcal{E}},x\rangle P_{(1,1,1)_{\mathcal{E}}}\left(\langle d(m,n),(0,0,1)_{\mathcal{E}}\rangle b + \langle b,(1,1,0)_{\mathcal{E}}\rangle d(m,n)\right)\\ &\quad = B(d(m,n)\cos\langle(1,1,0)_{\mathcal{E}},x\rangle + b\sin\langle(0,0,1)_{\mathcal{E}},x\rangle)\\ &\qquad + B(b\cos\langle(0,0,1)_{\mathcal{E}},x\rangle + d(m,n)\sin\langle(1,1,0)_{\mathcal{E}},x\rangle) \in E_2(\mathcal{W}).\end{aligned} \tag{2.14}$$

Let us define the set

$$\mathcal{G} := \{\langle d(m,n),(0,0,1)_{\mathcal{E}}\rangle b + \langle b,(1,1,0)_{\mathcal{E}}\rangle d(m,n) : b \in (0,0,1)_{\mathcal{E}}^{\perp}\}.$$

Since $\langle d(m,n), (0,0,1)_{\mathcal{E}}\rangle \neq 0$, $\mathcal{G}$ is a two-dimensional subspace of $\mathbb{R}^3$ contained in $(1,1,-1)_{\mathcal{E}}^{\perp}$. This shows that $\mathcal{G} = (1,1,-1)_{\mathcal{E}}^{\perp}$. Let us assume that

$$(1,1,1)_{\mathcal{E}} \notin (1,1,-1)_{\mathcal{E}}^{\perp}. \tag{2.15}$$

This assumption implies that the orthogonal projection $P_{(1,1,1)_{\mathcal{E}}}\mathcal{G}$ coincides with $(1,1,1)_{\mathcal{E}}^{\perp}$ and proves that $\mathcal{A}_{(1,1,1)_{\mathcal{E}}} \subset E_2(\mathcal{W})$. Similarly, one can show that $\mathcal{B}_{(1,1,1)_{\mathcal{E}}} \subset E_2(\mathcal{W})$. Finally, writing

$$(1,1,0)_{\mathcal{E}} = (1,1,1)_{\mathcal{E}} - (0,0,1)_{\mathcal{E}}$$

and applying the result of Step 1 to the set $\mathcal{W}_1 := \mathcal{W} \cup \{(1,1,1)_{\mathcal{E}}, (0,0,1)_{\mathcal{E}}\}$, we see that

$$\begin{aligned} d((1,1,1)_{\mathcal{E}}, (0,0,1)_{\mathcal{E}}) \cos\langle (1,1,0)_{\mathcal{E}}, x\rangle \in E_1(\mathcal{W}_1) = \mathcal{F}(E(\mathcal{W}_1)) \\ \subset \mathcal{F}(E_2(\mathcal{W})) = E_3(\mathcal{W}). \end{aligned}$$

Since $d((1,1,1)_{\mathcal{E}}, (0,0,1)_{\mathcal{E}}) \nparallel d(m,n)$, we get that $\mathcal{A}_{m+n} \subset E_3(\mathcal{W})$, under condition (2.15).

The same proof gives the result if at least one of the following conditions is satisfied: $(1,1,1)_{\mathcal{E}} \notin (1,-1,1)_{\mathcal{E}}^{\perp}$, $(1,1,1)_{\mathcal{E}} \notin (-1,1,1)_{\mathcal{E}}^{\perp}$.

Step 3. Let us assume now that

$$(1,1,1)_{\mathcal{E}} \in (1,1,-1)_{\mathcal{E}}^{\perp}, \quad (1,1,1)_{\mathcal{E}} \in (1,-1,1)_{\mathcal{E}}^{\perp}, \quad (1,1,1)_{\mathcal{E}} \in (-1,1,1)_{\mathcal{E}}^{\perp}. \tag{2.16}$$

Then $(1,1,1)_{\mathcal{E}} \in (1,1,-1)_{\mathcal{E}}^{\perp} = \mathcal{G}$, hence the projection $P_{(1,1,1)_{\mathcal{E}}}\mathcal{G}$ is a one-dimensional subspace of $\mathcal{G}$. Let $f \in P_{(1,1,1)_{\mathcal{E}}}\mathcal{G}$ be a unit vector. From the definition of $\mathcal{G}$ it follows that

$$f \notin (0,0,1)_{\mathcal{E}}^{\perp}. \tag{2.17}$$

By (2.14),

$$f \cos\langle (1,1,1)_{\mathcal{E}}, x\rangle \in E_2(\mathcal{W}).$$

In the same way, one proves that

$$f \sin\langle (1,1,1)_{\mathcal{E}}, x\rangle \in E_2(\mathcal{W}).$$

Now applying (2.13), we obtain for any $b \in (0,0,1)_{\mathcal{E}}^{\perp}$

$$\begin{aligned} &\cos\langle (1,1,0)_{\mathcal{E}}, x\rangle P_{(1,1,0)_{\mathcal{E}}} \left(\langle f, (0,0,1)_{\mathcal{E}}\rangle b - \langle b, (1,1,1)_{\mathcal{E}}\rangle f\right) \\ &\quad = B(f\cos\langle (1,1,1)_{\mathcal{E}}, x\rangle + b \sin\langle (0,0,1)_{\mathcal{E}}, x\rangle) \\ &\qquad + B(-b\cos\langle (0,0,1)_{\mathcal{E}}, x\rangle + f \sin\langle (1,1,1)_{\mathcal{E}}, x\rangle) \in E_3(\mathcal{W}). \end{aligned} \tag{2.18}$$

Since we have (2.17), the set

$$\tilde{\mathcal{G}} := \{\langle f, (0,0,1)_{\mathcal{E}}\rangle b - \langle b, (1,1,1)_{\mathcal{E}}\rangle f : b \in (0,0,1)_{\mathcal{E}}^{\perp}\}$$

is a two-dimensional subspace of $\mathbb{R}^3$ contained in $(1,1,2)_{\mathcal{E}}^{\perp}$. This shows that $\tilde{\mathcal{G}} = (1,1,2)_{\mathcal{E}}^{\perp}$. Let us assume that

$$(1,1,0)_{\mathcal{E}} \notin (1,1,2)_{\mathcal{E}}^{\perp}. \tag{2.19}$$

Then the orthogonal projection $P_{(1,1,0)_{\mathcal{E}}}\tilde{\mathcal{G}}$ coincides with $(1,1,0)_{\mathcal{E}}^{\perp}$, and we get that $\mathcal{A}_{(1,1,0)_{\mathcal{E}}}, \mathcal{B}_{(1,1,0)_{\mathcal{E}}} \subset E_3(\mathcal{W})$.

By symmetry, if $(1,0,1)_{\mathcal{E}} \notin (1,2,1)_{\mathcal{E}}^{\perp}$, then $\mathcal{A}_{(1,0,1)_{\mathcal{E}}}, \mathcal{B}_{(1,0,1)_{\mathcal{E}}} \subset E_3(\mathcal{W})$. Then by Step 1,

$$d((1,0,1)_{\mathcal{E}}, (0,1,0)_{\mathcal{E}}) \cos\langle (1,1,1)_{\mathcal{E}}, x\rangle \in E_4(\mathcal{W}).$$

It is easy to verify that $d((1,0,1)_{\mathcal{E}},(0,1,0)_{\mathcal{E}}) \notin \mathcal{G}$, thus $\mathcal{A}_{(1,1,1)_{\mathcal{E}}} \subset E_4(\mathcal{W})$ and $\mathcal{B}_{(1,1,1)_{\mathcal{E}}} \subset E_4(\mathcal{W})$. From the arguments of the last part of Step 2 it follows now that $\mathcal{A}_{m+n} \subset E_5(\mathcal{W})$. The case $(0,1,1)_{\mathcal{E}} \notin (2,1,1)_{\mathcal{E}}^{\perp}$ is similar.

Step 4. It remains to consider the case when (2.15) holds and

$$(1,1,0)_{\mathcal{E}} \in (1,1,2)_{\mathcal{E}}^{\perp}, \quad (1,0,1)_{\mathcal{E}} \in (1,2,1)_{\mathcal{E}}^{\perp}, \quad (0,1,1)_{\mathcal{E}} \in (2,1,1)_{\mathcal{E}}^{\perp}. \tag{2.20}$$

In fact (2.16) and (2.20) are incompatible. Indeed, (2.20) and (2.16) are equivalent to, respectively,

$$\|m+n\|^2+2\langle m+n,r\rangle = 0, \|m+r\|^2+2\langle m+r,n\rangle = 0, \|r+n\|^2+2\langle r+n,m\rangle = 0, \tag{2.21}$$

$$\|m+n\| = \|r\|, \quad \|m+r\| = \|n\|, \quad \|r+n\| = \|m\|. \tag{2.22}$$

Taking the sum of the three equalities in (2.21) and using (2.22), we get

$$\|m\|^2 + \|n\|^2 + \|r\|^2 + 4(\langle m,n\rangle + \langle m,r\rangle + \langle r,n\rangle) = 0. \tag{2.23}$$

On the other hand, (2.22) is equivalent to

$$\|m\|^2 + 2\langle m,n\rangle + \|n\|^2 = \|r\|^2, \quad \|m\|^2 + 2\langle m,r\rangle + \|r\|^2 = \|n\|^2,$$
$$\|r\|^2 + 2\langle r,n\rangle + \|n\|^2 = \|m\|^2.$$

Summing these equalities, we obtain

$$\|m\|^2 + \|n\|^2 + \|r\|^2 + 2(\langle m,n\rangle + \langle m,r\rangle + \langle r,n\rangle) = 0.$$

Comparing this with (2.23), we see that

$$\langle m,n\rangle + \langle m,r\rangle + \langle r,n\rangle = 0.$$

Again using (2.23), we get $m = n = r = 0$, which is a contradiction. This completes the proof of Proposition 2.8.

□

Proof of Theorem 2.6. Step 1. Let us show that

$$E(\mathbb{Z}^3_{\mathcal{K}}) \subset E_\infty(\mathcal{K}). \tag{2.24}$$

To this end, we introduce the sets

$$\mathcal{K}_0 := \mathcal{K}, \quad \mathcal{K}_j = \mathcal{K}_{j-1} \cup \{m \pm n : m,n \in \mathcal{K}_{j-1}, m \not\parallel n\}, \quad j \geq 1.$$

From Proposition 2.8 it follows that

$$\begin{aligned} E(\mathcal{K}_j) \subset E_5(\mathcal{K}_{j-1}) &= \mathcal{F}^5(E(\mathcal{K}_{j-1})) \subset \mathcal{F}^{10}(E(\mathcal{K}_{j-2})) \subset \ldots \subset \mathcal{F}^{5j}(E(\mathcal{K})) \\ &= E_{5j}(\mathcal{K}). \end{aligned} \tag{2.25}$$

On the other hand, since $\mathcal{K}$ is a generator of $\mathbb{Z}^3_{\mathcal{K}}$, one easily checks that $\cup_{j=1}^{\infty}\mathcal{K}_j = \mathbb{Z}^3_{\mathcal{K}}$. Combining this with (2.25), we get (2.24).

Step 2. Now let us prove that

$$E_\infty(\mathcal{K}) \subset E(\mathbb{Z}^3_{\mathcal{K}}). \tag{2.26}$$

For any $\eta_1 \in E_1(\mathcal{K}_{j-1})$ and $j \geq 1$, there are vectors $\eta, \zeta^1, \ldots, \zeta^p \in E(\mathcal{K}_{j-1})$ satisfying the relation

$$\eta_1 = \eta - \sum_{i=1}^{p} B(\zeta^i).$$

Here we use the following simple lemma.

Lemma 2.9. *For any $j \geq 1$, we have*

$$\{B(\zeta) : \zeta \in E(\mathcal{K}_{j-1})\} \subset E(\mathcal{K}_j).$$

This lemma implies that

$$E_1(\mathcal{K}_{j-1}) \subset E(\mathcal{K}_j).$$

Iterating this, we get

$$E_j(\mathcal{K}) \subset E(\mathcal{K}_j),$$

hence

$$E_\infty(\mathcal{K}) = \bigcup_{j=1}^{\infty} E_j(\mathcal{K}) \subset \bigcup_{j=1}^{\infty} E(\mathcal{K}_j) \subset E\left(\bigcup_{j=1}^{\infty} \mathcal{K}_j\right) = E(\mathbb{Z}^3_{\mathcal{K}}).$$

This proves (2.26) and (2.8).

Step 3. If $\mathcal{K}$ is a generator of $\mathbb{Z}^3$, then (2.8) implies that $E(\mathcal{K})$ is saturating in H^k_σ for any $k \geq 0$.

Now let us assume that $\mathcal{K}$ is not a generator of $\mathbb{Z}^3$, i.e., there is $\ell \in \mathbb{Z}^3$ such that $\ell \notin \mathbb{Z}^3_{\mathcal{K}}$. Then it follows from (2.8) that c_ℓ is orthogonal to $E_\infty(\mathcal{K})$ in H. This shows that $E(\mathcal{K})$ is not saturating in H and completes the proof of the theorem.

□

Proof of Lemma 2.9. For any $\zeta \in E(\mathcal{K}_{j-1})$, we have

$$\zeta = \sum_{\ell \in \pm\mathcal{K}_{j-1}} (a_\ell c_\ell + b_\ell s_\ell)$$

for some $a_\ell, b_\ell \in \mathbb{R}$. It follows that

$$B(\zeta) = \sum_{m,n \in \pm\mathcal{K}_{j-1}} (a_m a_n B(c_m, c_n) + b_m b_n B(s_m, s_n) \\ + a_m b_n B(c_m, s_n) + b_m a_n B(s_m, c_n)).$$

Using some trigonometric identities, it is easy to verify that

$$B(c_m, c_n) \in \operatorname{span}\{s_{m+n}, s_{m-n}\} \subset E(\mathcal{K}_j).$$

In a similar way, one gets $B(s_m, s_n), B(c_m, s_n), B(s_m, c_n) \in E(\mathcal{K}_j)$. □

For any finite set $\mathcal{K} \subset \mathbb{Z}^3$ and $k \geq 3$, let us define the space

$$H^k_{\sigma,\mathcal{K}} := \overline{E_\infty(\mathcal{K})}^{H^k}.$$

From the structure of the nonlinearity it follows that $H^k_{\sigma,\mathcal{K}}$ is invariant for (2.1) when $h, \eta \in L^2(J_T, H^{k-1}_{\sigma,\mathcal{K}})$. Moreover, $H^k_{\sigma,\mathcal{K}} = H^k_\sigma$ if and only if $\mathcal{K}$ is a generator of $\mathbb{Z}^3$. As a corollary we get the following characterisation of the controllability in H^k_σ.

Theorem 2.10. *Let $\mathcal{K} \subset \mathbb{Z}^3$ be a finite set and $h \in L^2(J_T, H^{k-1}_{\sigma,\mathcal{K}})$. Then equation (2.1) with $\eta \in C^\infty(J_T, E(\mathcal{K}))$ is controllable in the space H^k_σ at time T if and only if $\mathcal{K}$ is a generator of $\mathbb{Z}^3$.*

It is also interesting to study the controllability properties of the NS system when $E(\mathcal{K})$ given by (2.7) is not saturating (i.e., $\mathcal{K}$ is not a generator of $\mathbb{Z}^3$). Let us note that the space $E(\mathcal{K})$ is saturating in $H^k_{\sigma,\mathcal{K}}$ for any $\mathcal{K} \subset \mathbb{Z}^3$ and $k \geq 0$ (in the sense that $E_\infty(\mathcal{K})$ is dense in $H^k_{\sigma,\mathcal{K}}$). We have the following refined version of Theorem 2.2.

Theorem 2.11. *For any non-empty finite $\mathcal{K} \subset \mathbb{Z}^3$ and $h \in L^2(J_T, H^{k-1}_{\sigma,\mathcal{K}})$, equation (2.1) with $\eta \in C^\infty(J_T, E(\mathcal{K}))$ is controllable in the space $H^k_{\sigma,\mathcal{K}}$ at time T, i.e., for any $\varepsilon > 0$ and any*

$$\varphi \in C(J_T, H^k_{\sigma,\mathcal{K}}) \cap L^2(J_T, H^{k+1}_{\sigma,\mathcal{K}}) \cap W^{1,2}(J_T, H^{k-1}_{\sigma,\mathcal{K}})$$

there is a control $\eta \in \Theta(h, u_0) \cap C^\infty(J_T, E(\mathcal{K}))$ such that

$$\|\mathcal{R}_T(u_0, h+\eta) - \varphi(T)\|_k + \|\!|\mathcal{R}(u_0, h+\eta) - \varphi|\!\|_{T,k} + \|\phi^{\mathcal{R}(u_0,h+\eta)} - \phi^\varphi\|_{L^\infty(J_T,C^1)} < \varepsilon,$$

where $u_0 = \varphi(0)$.

The proof of this result literally repeats the arguments of the proof of Theorem 2.2, so we omit the details.

2.2.2 Controls with two vanishing components

In this section, we consider the NS system

$$\dot{u} - \nu\Delta u + \langle u, \nabla\rangle u + \nabla p = h(t,x) + (0,0,1)\eta(t,x), \quad \operatorname{div} u = 0, \tag{2.27}$$

$$u(0) = u_0, \tag{2.28}$$

where η is a control taking values in a finite-dimensional space of the form

$$\mathcal{H}(\mathcal{K}) := \operatorname{span}\{\cos\langle m, x\rangle, \sin\langle m, x\rangle : m \in \mathcal{K}\},$$

where $\mathcal{K}$ is a subset of $\mathbb{Z}^3$, and h is a given smooth divergence-free function. Let us rewrite (2.27) in an equivalent form

$$\dot{u} - \nu\Delta u + B(u) = h(t,x) + \tilde{\eta}(t,x), \tag{2.29}$$

where $\tilde{\eta} := \Pi(e\eta)$ and $e := (0,0,1)$. Then the control $\tilde{\eta}$ takes values in the space

$$\tilde{E}(\mathcal{K}) := \text{span}\{(P_m e)\cos\langle m, x\rangle, (P_m e)\sin\langle m, x\rangle : m \in \mathcal{K}\}. \tag{2.30}$$

For an appropriate choice of $\mathcal{K}$, this space is saturating.

Proposition 2.12. *Let*

$$\mathcal{K} := \{(1,0,0), (0,1,0), (1,0,1), (0,1,1)\}. \tag{2.31}$$

Then $\tilde{E}(\mathcal{K})$ is an 8-dimensional saturating space in H^k_σ for any $k \geq 0$.

Combining this proposition with Theorem 2.2, we get immediately the following result.

Theorem 2.13. *Let $h \in L^2(J_T, H^{k-1}_\sigma), k \geq 3$, and $T > 0$. If $\mathcal{K}$ is defined by (2.31), then system (2.29) with $\tilde{\eta} \in C^\infty(J_T, \tilde{E}(\mathcal{K}))$ is controllable at time T.*

Proof. Step 1. Let us first show that $\mathcal{A}_{(0,0,1)} \subset \mathcal{F}(\tilde{E}(\mathcal{K}))$. Using (2.13), we get for any $\lambda \in \mathbb{R}$

$$\begin{aligned}
&\lambda(-1/2,0,0)\cos\langle(0,0,1),x\rangle \\
&\qquad = B(\lambda(P_{(1,0,0)}e)\cos\langle(1,0,0),x\rangle + (P_{(1,0,1)}e)\sin\langle(1,0,1),x\rangle) \\
&\qquad\quad + B(-(P_{(1,0,1)}e)\cos\langle(1,0,1),x\rangle + \lambda(P_{(1,0,0)}e)\sin\langle(1,0,0),x\rangle), \\
&\lambda(0,-1/2,0)\cos\langle(0,0,1),x\rangle \\
&\qquad = B(\lambda(P_{(0,1,0)}e)\cos\langle(0,1,0),x\rangle + (P_{(0,1,1)}e)\sin\langle(0,1,1),x\rangle) \\
&\qquad\quad + B(-(P_{(0,1,1)}e)\cos\langle(0,1,1),x\rangle + \lambda(P_{(0,1,0)}e)\sin\langle(0,1,0),x\rangle).
\end{aligned}$$

The definition of $\mathcal{F}$ implies that $\mathcal{A}_{(0,0,1)} \subset \mathcal{F}(\tilde{E}(\mathcal{K}))$. A similar computation gives that $\mathcal{B}_{(0,0,1)} \subset \mathcal{F}(\tilde{E}(\mathcal{K}))$.

Step 2. Again using (2.13), we obtain for any $b := (b_1, b_2, 0) \in \mathbb{R}^3$

$$\begin{aligned}
(0, b_2/2, -b_1/2)\cos\langle(1,0,0),x\rangle &= B((P_{(1,0,1)}e)\cos\langle(1,0,1),x\rangle + b\sin\langle(0,0,1),x\rangle) \\
&\quad + B(-b\cos\langle(0,0,1),x\rangle + (P_{(1,0,1)}e)\sin\langle(1,0,1),x\rangle) \in \mathcal{F}^2(\tilde{E}(\mathcal{K})).
\end{aligned}$$

This shows that $\mathcal{A}_{(1,0,0)} \subset \mathcal{F}^2(\tilde{E}(\mathcal{K}))$. Similarly one proves also

$$\mathcal{B}_{(1,0,0)}, \mathcal{A}_{(0,1,0)}, \mathcal{B}_{(0,1,0)} \subset \mathcal{F}^2(\tilde{E}(\mathcal{K})).$$

Thus the result follows from the fact that $\{(1,0,0), (0,1,0), (0,0,1)\}$ is a generator of $\mathbb{Z}^3$.

□

2.2.3 6-dimensional exemple

The following result, combined with Theorem 2.2, shows that that the 3D NS system can be controlled with η taking values in a 6-dimensional space.

Proposition 2.14. *Let us define the following* 6*-dimensional space*

$$\begin{aligned}\hat{E} := \operatorname{span}\{&a\cos\langle(1,0,1),x\rangle, a\sin\langle(1,0,1),x\rangle,\\ &e\cos\langle(0,1,1),x\rangle, e\sin\langle(0,1,1),x\rangle,\\ &b\cos\langle(0,0,1),x\rangle, b\sin\langle(0,0,1),x\rangle\},\end{aligned} \tag{2.32}$$

where $a := (1,1,1), b := (1,0,0), e := (0,0,1)$*. Then* $\hat{E}$ *is saturating in* H^k_σ *for any* $k \geq 0$*.*

Proof. Step 1. Let us first show that $\mathcal{A}_{(1,-1,0)} \subset \mathcal{F}^2(\hat{E})$. Using (2.13), we get for any $\lambda \in \mathbb{R}$

$$\begin{aligned}\lambda(0,-1,-1)\cos\langle(1,0,0),x\rangle &= B(\lambda a\cos\langle(1,0,1),x\rangle + b\sin\langle(0,0,1),x\rangle)\\ &\quad + B(-b\cos\langle(0,0,1),x\rangle + \lambda a\sin\langle(1,0,1),x\rangle) \in \mathcal{F}(\hat{E}),\\ \lambda(1,0,0)\cos\langle(0,1,0),x\rangle &= B(\lambda e\cos\langle(0,1,1),x\rangle + b\sin\langle(0,0,1),x\rangle)\\ &\quad + B(-b\cos\langle(0,0,1),x\rangle + \lambda e\sin\langle(0,1,1),x\rangle) \in \mathcal{F}(\hat{E})\end{aligned}$$

and $(0,-1,-1)\sin\langle(1,0,0),x\rangle, (1,0,0)\sin\langle(0,1,0),x\rangle \in \mathcal{F}(\hat{E})$, similarly. Writing

$$(1,-1,0) = (1,0,0) - (0,1,0) = (1,0,1) - (0,1,1)$$

and applying (2.13), we see that

$$\begin{aligned}\lambda(0,0,1)\cos\langle(1,-1,0),x\rangle &= B(\lambda(0,-1,-1)\cos\langle(1,0,0),x\rangle + b\sin\langle(0,1,0),x\rangle)\\ &\quad + B(-b\cos\langle(0,1,0),x\rangle + \lambda(0,-1,-1)\sin\langle(1,0,0),x\rangle) \in \mathcal{F}^2(\hat{E}),\\ \lambda(-1,-1,1)\cos\langle(1,-1,0),x\rangle &= B(\lambda a\cos\langle(1,0,1),x\rangle + e\sin\langle(0,1,1),x\rangle)\\ &\quad + B(-e\cos\langle(0,1,1),x\rangle + \lambda a\sin\langle(1,0,1),x\rangle) \in \mathcal{F}^2(\hat{E}).\end{aligned}$$

This proves that $\mathcal{A}_{(1,-1,0)} \subset \mathcal{F}^2(\hat{E})$. A similar computation establishes that $\mathcal{B}_{(1,-1,0)} \subset \mathcal{F}^2(\hat{E})$.

Step 2. Let us show that $\mathcal{A}_{(1,0,0)}, \mathcal{B}_{(1,0,0)} \subset \mathcal{F}^3(\hat{E})$. Taking any vector $f := (f_1, f_1, f_2) \in (1,-1,0)^\perp$, we apply (2.12)

$$\begin{aligned}(0,f_1,f_2)\cos\langle(1,0,0),x\rangle &= B(f\cos\langle(1,-1,0),x\rangle + b\sin\langle(0,1,0),x\rangle)\\ &\quad + B(b\cos\langle(0,1,0),x\rangle + f\sin\langle(1,-1,0),x\rangle) \in \mathcal{F}^3(\hat{E}).\end{aligned}$$

This proves that $\mathcal{A}_{(1,0,0)} \subset \mathcal{F}^3(\hat{E})$, and $\mathcal{B}_{(1,0,0)} \subset \mathcal{F}^3(\hat{E})$ is similar.

Step 3. Let us show that $\mathcal{A}_{(0,0,1)}$. Again we shall prove only the first inclusion. For any $g := (0, g_1, g_2) \in \mathbb{R}^3$, we apply (2.13)

$$\begin{aligned}(-g_2, g_1 - g_2, 0)\cos\langle(0,0,1),x\rangle &= B(a\cos\langle(1,0,1),x\rangle + g\sin\langle(1,0,0),x\rangle)\\ &\quad + B(-g\cos\langle(1,0,0),x\rangle + a\sin\langle(1,0,1),x\rangle) \in \mathcal{F}^3(\hat{E}).\end{aligned}$$

This proves that $\mathcal{A}_{(0,0,1)} \subset \mathcal{F}^4(\hat{E})$ and $\mathcal{B}_{(0,0,1)} \subset \mathcal{F}^4(\hat{E})$ is similar. By Theorem 2.7, we have that the family $\{(1,0,0),(0,0,1),(1,-1,0)\}$ is a generator of $\mathbb{Z}^3$. Thus applying Theorem 2.6, we complete the proof.

□

It would be interesting to get a characterisation of finite-dimensional saturating spaces of the following general form

$$E(\mathcal{K}_c,\mathcal{K}_s,a,b) := \operatorname{span}\{a_m\cos\langle m,x\rangle; b_n\sin\langle n,x\rangle : m\in\mathcal{K}_c, n\in\mathcal{K}_s\},$$

where $\mathcal{K}_c,\mathcal{K}_s\subset\mathbb{Z}^3$, $a:=\{a_m\}_{m\in\mathcal{K}_c}\subset\mathbb{R}^3_*$, and $b:=\{b_n\}_{n\in\mathcal{K}_s}\subset\mathbb{R}^3_*$. From the results of Subsection 2.2.1 it follows that both $\mathcal{K}_c$ and $\mathcal{K}_s$ are necessarily generators of $\mathbb{Z}^3$.

3 Proof Theorem 2.5

The proof follows the arguments of [AS05, AS06], and [Shi06]. We consider the following system

$$\dot u + L(u+\zeta) + B(u+\zeta) = h+\eta \tag{3.1}$$

with two E-valued controls η,ζ. We denote by $\hat\Theta(u_0,h)$ the set of $(\eta,\zeta)\in L^2(J_T,H^{k-1}_\sigma)\times L^4(J_T,H^{k+1}_\sigma)$ for which problem (3.1), (0.2) has a solution in $\mathcal{X}_{T,k}$. Theorem 2.5 is deduced from the following proposition which is proved at the end of this section (cf. Proposition 3.2 in [Shi06]).

Proposition 3.1. *For any $\eta_1\in\Theta(u_0,h)\cap L^2(J_T,E_1)$, there is a sequence $(\eta_n,\zeta_n)\in\hat\Theta(u_0,h)\cap C^\infty(J_T,E\times E)$ such that*

$$\|\mathcal{R}(u_0,h+\eta_1)-\mathcal{R}(u_0,h+\eta_n,\zeta_n)\|_{L^\infty(J_T,H^k)}+|||\zeta_n|||_{T,k}\to 0 \quad \textit{as } n\to\infty, \tag{3.2}$$

$$\sup_{n\ge1}(\|\mathcal{R}(u_0,h+\eta_n,\zeta_n)\|_{\mathcal{X}_{T,k}}+\|\zeta_n\|_{L^\infty(J_T,H^{k+1})}+\|\eta_n\|_{L^2(J_T,H^{k-1})})<\infty. \tag{3.3}$$

Proof of Theorem 2.5. Let us take any $u_0\in H^k_\sigma$ and $\eta_1\in\Theta(h,u_0)\cap L^2(J_T,E_1)$, and let $(\eta_n,\zeta_n)\in\hat\Theta(u_0,h)\cap C^\infty(J_T,E\times E)$ be any sequence satisfying (3.2) and (3.3). Let $\hat\zeta_n\in C^\infty(J_T,E)$ be such that $\hat\zeta_n(0)=\hat\zeta_n(T)=0$ and

$$\begin{aligned}&\|\zeta_n-\hat\zeta_n\|_{L^4(J_T,H^{k+1})}\to 0 \text{ as } n\to\infty,\\ &\sup_{n\ge1}\|\hat\zeta_n\|_{L^\infty(J_T,H^{k+1})}<+\infty.\end{aligned} \tag{3.4}$$

By Theorem 1.3 and (3.3), for sufficiently large $n\ge1$, we have $(\eta_n,\hat\zeta_n)\in\hat\Theta(u_0,h)$ and

$$\|\mathcal{R}(u_0,h+\eta_n,\zeta_n)-\mathcal{R}(u_0,h+\eta_n,\hat\zeta_n)\|_{\mathcal{X}_{T,k}}\to 0 \quad \text{as } n\to\infty. \tag{3.5}$$

Notice that

$$\begin{aligned}\mathcal{R}_t(u_0,h+\eta_n,\hat\zeta_n)&=\mathcal{R}_t(u_0,h+\hat\eta_n)-\hat\zeta_n(t) \quad \text{for } t\in J_T,\\ \mathcal{R}_T(u_0,h+\eta_n,\hat\zeta_n)&=\mathcal{R}_T(u_0,h+\hat\eta_n),\end{aligned} \tag{3.6}$$

where $\hat{\eta}_n := \eta + \dot{\hat{\zeta}}_n$. Thus (3.5) implies that

$$\|\mathcal{R}_T(u_0, h+\eta_1) - \mathcal{R}_T(u_0, h+\hat{\eta}_n)\|_k \to 0 \quad \text{as } n \to \infty.$$

Using (3.2), (3.5), (3.6), and the fact that

$$|||\hat{\zeta}_n|||_{T,k} \to 0 \quad \text{as } n \to \infty, \tag{3.7}$$

we obtain

$$\begin{aligned}
&|||\mathcal{R}(u_0, h+\eta_1) - \mathcal{R}(u_0, h+\hat{\eta}_n)|||_{T,k} \\
&\qquad \le T\|\mathcal{R}(u_0, h+\eta_1) - \mathcal{R}(u_0, h+\eta_n, \zeta_n)\|_{L^\infty(J_T, H^k)} \\
&\qquad\quad + T\|\mathcal{R}(u_0, h+\eta_n, \zeta_n) - \mathcal{R}(u_0, h+\eta_n, \hat{\zeta}_n)\|_{\mathcal{X}_{T,k}} \\
&\qquad\quad + |||\mathcal{R}(u_0, h+\eta_n, \hat{\zeta}_n) - \mathcal{R}(u_0, h+\hat{\eta}_n)|||_{T,k} \\
&\qquad \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Combining this with the embedding $H^3 \subset C^{1,1/2}$, (3.3), (3.6), and applying Lemma 1.1 with $\lambda = 1/2$, we get that

$$\|\phi^{\mathcal{R}(u_0, h+\eta_1)} - \phi^{\mathcal{R}(u_0, h+\hat{\eta}_n)}\|_{L^\infty(J_T, C^1)} \to 0 \quad \text{as } n \to \infty.$$

This completes the proof of Theorem 2.5. □

Proof of Proposition 3.1. Step 1. Without loss of generality, we can assume that $\eta_1 \in \Theta(u_0, h) \cap E_1$ is constant. Indeed, the general case is then obtained by approximating with piecewise constant controls and successive applications of the result on the intervals of constancy.

By the definition of $\mathcal{F}(E)$, for any $\eta_1 \in E_1$, there are vectors $\xi^1, \dots, \xi^n, \eta \in E$ such that

$$\eta_1 = \eta - \sum_{i=1}^{n} B(\xi^i).$$

Choosing $m = 2n$ and

$$\zeta^i := -\zeta^{i+n} := \frac{1}{\sqrt{2}} \xi^i, \quad i = 1, \dots, n,$$

it is easy to see that

$$B(u) - \eta_1 = \frac{1}{m} \sum_{j=1}^{m} \left(B(u+\zeta^j) + L\zeta^j\right) - \eta \quad \text{for any } u \in H^1_\sigma. \tag{3.8}$$

Then $u_1 := \mathcal{R}(u_0, h+\eta_1) \in \mathcal{X}_{T,k}$ satisfies the following equation

$$\dot{u}_1 + Lu_1 + \frac{1}{m} \sum_{j=1}^{m} \left(B(u+\zeta^j) + L\zeta^j\right) = h(t) + \eta. \tag{3.9}$$

Let us define $\zeta_n(t) = \zeta(\frac{nt}{T})$, where $\zeta(t)$ is a 1-periodic function such that

$$\zeta(s) = \zeta^j \text{ for } s \in [(j-1)/m, j/m)\,, j = 1, \ldots, m.$$

Equation (3.9) is equivalent to

$$\dot{u}_1 + L(u_1 + \zeta_n) + B(u_1 + \zeta_n) = h(t) + \eta + f_n(t),$$

where

$$f_n(t) := L\zeta_n + B(u_1 + \zeta_n) - \frac{1}{m}\sum_{j=1}^{m}\left(B(u_1 + \zeta^j) + L\zeta^j\right). \tag{3.10}$$

For any $f \in L^2(J_T, H)$, let us set

$$Kf(t) = \int_0^t e^{-(t-s)L} f(s)\mathrm{d}s.$$

It is easy to check that

$$K \text{ is continuous from } L^2(J_T, H_\sigma^{p-1}) \text{ to } \mathcal{X}_{T,p} \text{ for any } p \geq 1, \tag{3.11}$$

and $v_n = u_1 - Kf_n$ is a solution of the problem

$$\begin{aligned} \dot{v}_n + L(v_n + \zeta_n) + B(v_n + \zeta_n + Kf_n) &= h(t) + \eta, \\ v_n &= u_0. \end{aligned} \tag{3.12}$$

Step 2. Let us show that

$$\|Kf_n\|_{L^\infty(J_T, H^k)} \to 0 \quad \text{as } n \to \infty. \tag{3.13}$$

Indeed, the definition of ζ_n gives that

$$\sup_{n\geq 1} \|\zeta_n\|_{L^\infty(J_T, H^{k+1})} < \infty. \tag{3.14}$$

Combining this with (3.10), (1.14), and the fact that $u_1 \in \mathcal{X}_{T,k}$, we get

$$\sup_{n\geq 1} \|f_n\|_{L^\infty(J_T, H^{k-1})} < \infty. \tag{3.15}$$

This implies that

$$\begin{aligned} \sup_{n\geq 1} \|Kf_n\|_{L^\infty(J_T, H_\sigma^{k+1/2})} &\leq C \sup_{n\geq 1, t\in[0,T]} \int_0^t \|L^{3/4} e^{-(t-s)L}\|_{\mathcal{L}(H)} \|f_n(s)\|_{k-1}\mathrm{d}s \\ &\leq C_1 \sup_{n\geq 1, t\in[0,T]} \int_0^t (t-s)^{-3/4} \|f_n(s)\|_{k-1}\mathrm{d}s \\ &\leq C_2 \sup_{n\geq 1} \|f_n\|_{L^\infty(J_T, H^{k-1})} < \infty, \end{aligned} \tag{3.16}$$

where we used the inequality

$$\|L^r e^{-tL}\|_{\mathcal{L}(H)} \leq C_3 t^{-r} \quad \text{for any } r \geq 0, t > 0.$$

In Step 4 of the proof of Proposition 3.2 in [Shi06], it is established that

$$\|Kf_n\|_{L^\infty(J_T, H^1)} \to 0.$$

Using this with (3.16) and an interpolation inequality, we get (3.13). Combining (3.11) with (3.15), we obtain also that

$$\sup_{n\geq 1} \|Kf_n\|_{\mathcal{X}_{T,k}} < \infty. \tag{3.17}$$

Step 3. Equation (3.12) can be rewritten as

$$\dot{v}_n + L(v_n + \zeta_n) + B(v_n + \zeta_n) = h(t) + \eta + g_n(t), \tag{3.18}$$

where

$$g_n(t) := -(B(v_n + \zeta_n, Kf_n) + B(Kf_n, v_n + \zeta_n) + B(Kf_n)).$$

From (3.13), (3.3), and (1.14) it is easy to deduce that $\|g_n\|_{L^2(J_T, H^{k-1})} \to 0$ as $n \to \infty$. From (3.17) it follows that

$$\sup_{n\geq 1} \|v_n\|_{\mathcal{X}_{T,k}} < \infty.$$

Therefore, by Theorem 1.3 and (3.14), we have $(\eta, \zeta_n) \in \hat{\Theta}(u_0, h)$ for sufficiently large $n \geq 1$ and

$$\|\mathcal{R}(u_0, \zeta_n, \eta) - v_n\|_{\mathcal{X}_{T,k}} \to 0 \text{ as } n \to \infty.$$

On the other hand, by (3.13),

$$\|v_n - u_1\|_{L^\infty(J_T, H^k)} \to 0 \text{ as } n \to \infty,$$

whence

$$\|\mathcal{R}(u_0, \zeta_n, \eta) - u_1\|_{L^\infty(J_T, H^k)} \to 0 \text{ as } n \to \infty,$$
$$\sup_{n\geq 1} \|\mathcal{R}(u_0, \zeta_n, \eta)\|_{\mathcal{X}_{T,k}} < +\infty.$$

Step 4. Let us show that

$$|||\zeta_n|||_{T,k} \to 0 \quad \text{as } n \to \infty. \tag{3.19}$$

We set $\mathcal{L}\zeta_n(t) := \int_0^t \zeta_n(s)\mathrm{d}s$. It suffices to check that

(i) the sequence $\mathcal{L}\zeta_n$ is relatively compact in $C(J_T, H^k_\sigma)$.

(ii) for any $t \in J_T$, $\mathcal{L}\zeta_n(t) \to 0$ in H^k_σ as $n \to \infty$.

To prove the first assertion, we use the Arzelà–Ascoli theorem. The functions ζ_n are piecewise constant and the set $\zeta_n(t), t \in J_T$ is contained in a finite subset of H^{k+1}_σ not depending on n. This implies that there is a compact set $F \subset H^{k+1}_\sigma$ such that

$$\mathcal{L}\zeta_n(t) \in F \text{ for all } t \in J_T, n \geq 1.$$

From (3.3) it follows that the sequence $\mathcal{L}\zeta_n$ is uniformly equicontinuous on J_T. Thus, by the Arzelà–Ascoli theorem, $\mathcal{L}\zeta_n$ is relatively compact in $C(J_T, H^k_\sigma)$.

Let us prove (ii). Let $t = t_l + \tau$, where $t_l = \frac{lT}{n}$, $l \in \mathbb{N}$ and $\tau \in [0, \frac{T}{n})$. In view of the construction of ζ_n, we have that $\mathcal{L}\zeta_n(lT/n) = 0$. Combining this with (3.3), we get

$$\mathcal{L}\zeta_n(t) = \int_{\frac{lT}{n}}^t \zeta_n(s)\mathrm{d}s \to 0,$$

which completes the proof of (3.19).

Finally, taking an arbitrary sequence $\hat{\zeta}_n \in C^\infty(J_T, E)$ such that

$$\|\zeta_n - \hat{\zeta}_n\|_{L^\infty(J_T,E)} \to 0 \text{ as } n \to \infty,$$

and using Theorem 1.3, we see that the conclusions of Proposition 3.1 hold for the sequence $(\eta, \hat{\zeta}_n) \in C^\infty(J_T, E \times E)$.

□

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