Multiplicative ergodic theorem for a non-irreducible random dynamical system

D. Martirosyan* V. Nersesyan†

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Abstract

We study the asymptotic properties of the trajectories of a discrete-time random dynamical system in an infinite-dimensional Hilbert space. Under some natural assumptions on the model, we establish a multiplicative ergodic theorem with an exponential rate of convergence. The assumptions are satisfied for a large class of parabolic PDEs, including the 2D Navier–Stokes and complex Ginzburg–Landau equations perturbed by a non-degenerate bounded random kick force. As a consequence of this ergodic theorem, we derive some new results on the statistical properties of the trajectories of the underlying random dynamical system. In particular, we obtain large deviations principle for the occupation measures and the analyticity of the pressure function in a setting where the system is not irreducible. The proof relies on a refined version of the uniform Feller property combined with some contraction and bootstrap arguments.

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0 Introduction

This paper continues the study, initiated in [14], of the multiplicative ergodicity for the following discrete-time random dynamical system

$$u_k = S(u_{k-1}) + \eta_k, \quad k \geq 1. \tag{0.1}$$

Here $\{\eta_k\}$ is a sequence of bounded independent identically distributed (i.i.d) random variables in a separable Hilbert space H and $S : H \to H$ is a continuous mapping subject to some natural assumptions. Without going into formalities

*Inria of Paris, 2 rue Simone Iff, 75012 Paris, France; e-mail: Martirosyan.Davit@gmail.com

†Laboratoire de Mathématiques, UMR CNRS 8100, UVSQ, Université Paris-Saclay, 45 Av. des Etats-Unis, 78035, Versailles, France; e-mail: Vahagn.Nersesyan@math.uvsq.fr

here, let us only mention that these assumptions ensure that S is locally Lipschitz possessing some dissipative and regularising properties, and that the random perturbation $\{\eta_k\}$ is non-degenerate (see Conditions (A)-(D) in the next section). A large class of dissipative PDEs with some discrete random perturbation can be written in the form (0.1). This class includes the 2D Navier–Stokes and the complex Ginzburg–Landau equations (see Section 5.5).

System (0.1) defines a homogeneous family of Markov chains $(u_k, \mathbb{P}_u)$ parametrised by the initial condition. It is well known that this family admits a unique stationary measure μ which is *exponentially mixing* in the sense that for any 1-Lipschitz function $f : H \to \mathbb{R}$ and any initial condition $u_0 = u \in H$, we have

$$\left| \mathbb{E}_u f(u_k) - \int_H f(v)\mu(\mathrm{d}v) \right| \leq C(1 + \|u\|)e^{-\alpha k} \quad k \geq 0, \tag{0.2}$$

where C and α are some positive numbers. We refer the reader to the papers [7, 20, 6, 1] for the first results of this type and Chapter 3 of the book [21] for details on the problem of ergodicity for (0.1).

In this paper, motivated by applications to *large deviations*, we consider the asymptotic behavior of the product

$$\Xi_k^V f = f(u_k) \exp\left(\sum_{n=1}^{k} V(u_n) \right), \tag{0.3}$$

where $V : H \to \mathbb{R}$ is a given bounded Lipschitz-continuous function (potential). We say that $(u_k, \mathbb{P}_u)$ satisfies a *multiplicative ergodic theorem* (MET) if we have a limit of the form

$$\lambda_V^{-k} \mathfrak{P}_k^V f(u) \to h_V(u) \int_H f(v)\mu_V(\mathrm{d}v) \qquad \text{as } k \to \infty \tag{0.4}$$

for some number $\lambda_V > 0$, positive continuous function $h_V : H \to \mathbb{R}$, and a Borel probability measure μ_V not depending on f and u, where $\mathfrak{P}_k^V$ is the *Feynman–Kac semigroup* associated with potential V, that is $\mathfrak{P}_k^V f(u) = \mathbb{E}_u \left\{ \Xi_k^V f \right\}$. This is a form of Oseledec's theorem obtained in [26] (see also [27, 24, 22]).

Let us recall that, under Conditions (A)-(D), a MET is established in [14], in the case when the initial point u belongs to the support $\mathcal{A}$ of the stationary measure μ. The set $\mathcal{A}$ is compact in H and it is the smallest invariant set for system (0.1). We extend this result in three directions. Our first main result shows that under the same hypotheses, convergence (0.4) is exponential for $u \in \mathcal{A}$. We next show that if the oscillation of the potential V is sufficiently small, then the convergence holds true for any $u \in H$. Moreover, the rate of convergence is again exponential. Finally, our third result proves that this restriction on the oscillation of V can be dropped if the operator S is a subcontraction [1] with respect to a certain metric whose topology is weaker than the natural one on H.

[1] See Condition (E) in the next section for the definition and Section 5.5 for examples.

We provide a section with different applications. Probably the most important one is the large deviations principle (LDP). Our multiplicative ergodic theorems allow to prove the existence and analyticity of the pressure function. Combining this with the Gärtner–Ellis theorem, the Kifer's criterion (see [18]), and some techniques developed in [14], we obtain new local and global large deviations results for the occupation measures in a *non-irreducible*[2] setting. Previously, the LDP for randomly forced PDEs has been studied in the papers [9, 8, 25], in the continuous-time, and in [14, 13, 15], for the discrete-time case. Let us also mention the papers [29, 19], where the LDP is derived from the MET for Markov processes possessing the strong Feller property. In all these references the underlying Markov processes are *irreducible*.

We also give two other applications related to the random time in the strong law of large numbers and the speed of attraction of the support of the stationary measure.

Let us outline some ideas behind the proof of the multiplicative ergodicity. From the boundedness of the random variables η_k it follows that system (0.1) has a compact invariant absorbing set X. This allows to reduce our study to a compact phase space X. The function h_V and the measure μ_V are the eigenvectors of the Feynman–Kac semigroup corresponding to an eigenvalue λ_V, i.e.,

$$\mathfrak{P}_1^V h_V = \lambda_V h_V, \ \mathfrak{P}_1^* \mu_V = \lambda_V \mu_V,$$

where $\mathfrak{P}_1^{V*}$ is the dual of $\mathfrak{P}_1^V$. By normalising $\mathfrak{P}_k^V$, we reduce the problem to the study of the exponential mixing for the following auxiliary Markov semigroup:

$$\mathscr{S}_k^V g = \lambda_V^{-k} h_V^{-1} \mathfrak{P}_k^V (g h_V), \qquad g \in C(X).$$

The latter is achieved by showing that, for sufficiently large time k, the dual operator $\mathscr{S}_k^{V*}$ is a contraction in the space of Borel probability measures endowed with the dual-Lipschitz norm. The proof of the contraction relies on the following four ingredients (see Theorem 2.1): (i) *refined uniform Feller property*, (ii) *uniform irreducibility on* $\mathcal{A}$, (iii) *concentration near* $\mathcal{A}$, (iv) *exponential bound*. The first one is an enhanced version of the uniform Feller property from [20, 14] with specified constants. Its proof relies on the coupling method. In the particular case $V = 0$, it coincides with the *asymptotic strong Feller property* in [10, 11]. Property (ii) follows from the dissipativity of the system and is well known. The lack of irreducibility on the set $X \setminus \mathcal{A}$ is compensated by (iii) and (iv). Their verification is highly non-trivial, especially when there is no restriction on the oscillation of the potential V. We use the subcontraction condition on S and a bootstrap argument to derive them for (0.1).

Let us note that the proof of the existence of an eigenvector μ_V is standard, it follows by a simple application of the Leray–Schauder fixed point theorem. On the other hand, the existence of h_V is more delicate. It is derived from the above-mentioned properties (i), (ii), and (iv).

[2]Recall that $(u_k, \mathbb{P}_u)$ is irreducible, if for any $u \in H$ and any ball $B \subset H$, we have $\mathbb{P}_u\{u_l \in B\} > 0$ for some $l \geq 1$.

Finally, let us mention that the long-time behavior of the Feynman–Kac semigroup has been studied in the literature in the case when the Markov process is strong Feller and the space of probability measures is endowed with the total variation metric (e.g., see [29, 3]). Obviously, these results cannot be applied in our setting.

The paper is organised as follows. In Section 1, we formulate our main results on the multiplicative ergodicity. In Section 2, we establish an abstract exponential convergence criterion for generalised Markov semigroups, which is applied in Section 3 to prove the main results. Section 4 is devoted to the proof of the refined uniform Feller property. In Section 5, we present the above-mentioned applications of the MET and discuss some examples of PDEs verifying Conditions (A)-(E).

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Notation

For a Polish space (H, d), we shall use the following standard notation.

$C(H)$ is the space of continuous functions $f : H \to \mathbb{R}$. For any $X \subset H$ and $f \in C(H)$, we denote $\|f\|_X = \sup_{u\in X} |f(u)|$ and write $\|f\|_\infty$ instead of $\|f\|_H$.

$C_b(H)$ is the space of bounded functions $f \in C(H)$ with the norm $\|f\|_\infty$.

$L_b(H)$ is the space of functions $f \in C_b(H)$ for which the following norm is finite

$$\|f\|_L = \|f\|_\infty + \sup_{u\neq v} \frac{|f(u) - f(v)|}{d(u, v)}.$$

$\mathcal{M}_+(H)$ is the set of non-negative finite Borel measures on H endowed with the dual-Lipschitz metric

$$\|\mu_1 - \mu_2\|_L^* = \sup_{\|f\|_L\leq 1} |\langle f, \mu_1\rangle - \langle f, \mu_2\rangle|, \quad \mu_1, \mu_2 \in \mathcal{M}_+(H),$$

where $\langle f, \mu\rangle = \int_H f(u)\mu(\mathrm{d}u)$. $\mathcal{P}(H)$ is the subset of Borel probability measures.

$\mathrm{Osc}(f)$ is the oscillation of a function $f : H \to \mathbb{R}$, that is the number defined by $\sup_{u\in H} f(u) - \inf_{u\in H} f(u)$.

$B_R(a)$ is the closed ball in H of radius R centered at a. When H is Banach and $a = 0$, we write B_R instead of $B_R(0)$.

1 Main results

We consider problem (0.1) in a separable Hilbert space H endowed with the scalar product $(\cdot,\cdot)$ and the associated norm $\|\cdot\|$. We assume that the following hypotheses are satisfied for the mapping $S : H \to H$ and the sequence $\{\eta_k\}$; they are exactly the same as in [14]:

(A) *S is locally Lipschitz in H, and for any $R > r > 0$, there is a positive number $a < 1$ and an integer $n_0 \geq 1$ such that*

$$\|S^n(u)\| \leq a\|u\| \vee r \qquad \text{for } u \in B_R,\ n \geq n_0, \tag{1.1}$$

where S^n stands for the n^{th} iteration of S.

For any set $B \subset H$, let us define the sequence of sets

$$\mathcal{A}(0,B) = B, \quad \mathcal{A}(k,B) = S(\mathcal{A}(k-1,B)) + \mathcal{K}, \quad k \geq 1,$$

where $\mathcal{K}$ is the support of the law of η_1. We denote by $\mathcal{A}(B)$ the closure in H of the union of the sets $\mathcal{A}(k,B), k \geq 1$ and call it *the domain of attainability from B*.

(B) *There is a number $\rho > 0$ and a continuous function $k_0 = k_0(R)$ such that*

$$\mathcal{A}(k, B_R) \subset B_\rho \quad \text{for } R \geq 0,\ k \geq k_0(R). \tag{1.2}$$

(C) *There is an orthonormal basis $\{e_j\}$ in H such that, for any $R > 0$,*

$$\|(I - \mathsf{P}_N)(S(u_1) - S(u_2))\| \leq \gamma_N(R)\|u_1 - u_2\|, \quad u_1, u_2 \in B_R, \tag{1.3}$$

where P_N is the orthogonal projection onto $\mathrm{span}\{e_1,\dots,e_N\}$, *and $\gamma_N(R) \downarrow 0$ as $N \to \infty$.*

(D) *The random variable η_k is of the form $\eta_k = \sum_{j=1}^\infty b_j \xi_{jk} e_j$, where $\{e_j\}$ is the orthonormal basis from* (C), *$b_j > 0$ are numbers such that $\sum_{j=1}^\infty b_j^2 < \infty$, and ξ_{jk} are independent scalar random variables with law having a density p_j with respect to the Lebesgue measure. We assume that p_j is a continuously differentiable function such that $p_j(0) > 0$ and with support in $[-1,1]$.*

The reader is referred to the beginning of Section 3 for some comments about these conditions and to Section 5.5 for examples.

Let $\mathcal{A} = \mathcal{A}(\{0\})$ be the domain of attainability from zero. This set is compact in H and invariant for (0.1), i.e., for any $u_0 \in \mathcal{A}$, we have $u_k \in \mathcal{A}$ for all $k \geq 1$ almost surely. Recall that $\mathfrak{P}_k^V : C_b(H) \to C_b(H)$ is the Feynman–Kac semigroup associated with $V \in L_b(H)$, that is,

$$\mathfrak{P}_k^V f(u) = \mathbb{E}_u \left\{\Xi_k^V f\right\} = \mathbb{E}_u \left\{ f(u_k) \exp\left(\sum_{n=1}^k V(u_n)\right)\right\}$$

and $\mathfrak{P}_k^{V*} : \mathcal{M}_+(H) \to \mathcal{M}_+(H)$ stands for its dual. The following theorem is our first result.

Theorem 1.1. *Under Conditions* (A)-(D)*, system* (0.1) *is multiplicatively ergodic on $\mathcal{A}$ with any potential $V \in L_b(\mathcal{A})$, i.e., the following two assertions hold.*

Existence. *There is a number $\lambda_V > 0$, a measure $\mu_V \in \mathcal{P}(\mathcal{A})$ whose support coincides with $\mathcal{A}$, and a positive function $h_V \in L_b(\mathcal{A})$ such that for any $u \in \mathcal{A}$,*

$$\mathfrak{P}_1^V h_V(u) = \lambda_V h_V(u), \quad \mathfrak{P}_1^{V*} \mu_V = \lambda_V \mu_V. \tag{1.4}$$

Exponential convergence. *There are positive numbers γ_V and C_V such that*

$$\left|\lambda_V^{-k} \mathfrak{P}_k^V f(u) - \langle f, \mu_V \rangle h_V(u)\right| \le C_V e^{-\gamma_V k}, \qquad k \ge 1 \tag{1.5}$$

for any $u \in \mathcal{A}$ and $f \in L_b(\mathcal{A})$ with $\|f\|_L \le 1$.

The second result establishes multiplicative ergodicity on the whole space H, under the restriction that the oscillation of the potential is small.

Theorem 1.2. *Under Conditions* (A)-(D)*, there is a number $\delta > 0$ such that system* (0.1) *is multiplicatively ergodic on H with any potential $V \in L_b(H)$ satisfying* $\mathrm{Osc}(V) \le \delta$*. Namely, we have the following.*

Existence. *There is a number $\lambda_V > 0$, a measure $\mu_V \in \mathcal{P}(H)$ whose support coincides with $\mathcal{A}$, and a positive function $h_V \in C(H)$ such that* (1.4) *holds for any $u \in H$.*

Exponential convergence. *There is $\gamma_V > 0$ such that for any $R > 0$, we have inequality* (1.5) *for any $u \in B_R$, $f \in L_b(H)$ with $\|f\|_L \le 1$, and some number $C_V(R) > 0$.*

Our third result shows that the restriction on the smallness of $\mathrm{Osc}(V)$ in this theorem can be removed if we assume additionally that S is a subcontraction with respect to some metric.

(E) *There is a translation invariant*[3] *metric d' on H whose topology is weaker than the natural one on H such that*

$$d'(S(u), S(v)) \le d'(u, v) \qquad \text{for } u, v \in \mathcal{A}(B_{\rho+1}), \tag{1.6}$$

where ρ is the number in (B)*.*

Theorem 1.3. *Under Conditions* (A)-(E)*, system* (0.1) *is multiplicatively ergodic on H with any potential $V \in L_b(H)$.*

These three theorems are the main results of this paper. Their proof is based on an abstract exponential convergence result presented in the next section.

[3] i.e., $d'(u+w, v+w) = d'(u, v)$ for any $u, v, w \in H$.

2 Exponential convergence for generalised Markov semigroups

In Theorem 2.1 in [14], a convergence result is established for generalised Markov semigroups in a compact metric space (X,d). In this section, we extend that result by proving exponential convergence, under some natural additional hypotheses. Moreover, we relax the irreducibility condition which now holds only on some subset.

Recall that a family $\{P(u,\cdot), u \in X\} \subset \mathcal{M}_+(X)$ is a *generalised Markov kernel* if the function $u \mapsto P(u,\cdot)$ from X to $\mathcal{M}_+(X)$ is continuous and non-vanishing. To any such kernel we associate semigroups $\mathfrak{P}_k : C(X) \to C(X)$ and $\mathfrak{P}_k^* : \mathcal{M}_+(X) \to \mathcal{M}_+(X)$ by

$$\mathfrak{P}_k f(u) = \int_X P_k(u, \mathrm{d}v) f(v), \quad \mathfrak{P}_k^* \mu(\Gamma) = \int_X P_k(u,\Gamma)\mu(\mathrm{d}u),$$

where $P_k(u,\cdot)$ are the iterations of $P(u,\cdot)$.

Theorem 2.1. *Let $\{P(u,\cdot), u \in X\}$ be a generalised Markov kernel and $\mathcal{A}$ a closed set in X such that $P(u, X \setminus \mathcal{A}) = 0$ for $u \in \mathcal{A}$. Assume that the following conditions are satisfied.*

(i) Refined uniform Feller property. *For any $c \in (0,1)$, there is $C > 0$ such that for any $f \in L_b(X)$ and $v, v' \in X$, we have*

$$|\mathfrak{P}_k f(v) - \mathfrak{P}_k f(v')| \leq (C \|f\|_X + c\|f\|_L) \|\mathfrak{P}_k \mathbf{1}\|_X \, d(v,v'). \tag{2.1}$$

(ii) Uniform irreducibility on $\mathcal{A}$. *For any $r > 0$, there is an integer $m \geq 1$ and a number $p > 0$ such that*

$$P_m(u, B_r(\hat{u})) \geq p \quad \text{for all } u \in X \text{ and } \hat{u} \in \mathcal{A}.$$

Then there are positive numbers λ, γ, C, a measure $\mu \in \mathcal{P}(X)$ whose support coincides with $\mathcal{A}$, and a positive function $h \in L_b(\mathcal{A})$ such that for any $u \in \mathcal{A}$ and $\nu \in \mathcal{P}(\mathcal{A})$, we have

$$\mathfrak{P}_1 h(u) = \lambda h(u), \quad \mathfrak{P}_1^* \mu = \lambda \mu, \tag{2.2}$$

$$\|\lambda^{-k} \mathfrak{P}_k^* \nu - \langle h, \nu \rangle \mu\|_L^* \leq C e^{-\gamma k}. \tag{2.3}$$

Furthermore, let us assume additionally the following conditions.

(iii) Concentration near $\mathcal{A}$. *The following limit holds*

$$\lim_{k\to\infty} \|P_k(\cdot, X \setminus \mathcal{A}_r)\|_X = 0 \qquad \text{for any } r > 0, \tag{2.4}$$

where $\mathcal{A}_r$ is the r-neighborhood of $\mathcal{A}$:

$$\mathcal{A}_r = \left\{ u \in X : \inf_{v \in \mathcal{A}} d(u,v) < r \right\}. \tag{2.5}$$

(iv) Exponential bound. *We have*

$$\Lambda = \sup_{k\geq 1} \lambda^{-k} \|\mathfrak{P}_k \mathbf{1}\|_X < \infty. \tag{2.6}$$

Then h has a positive Lipschitz-continuous extension to the space X (again denoted by h) such that (2.2) and (2.3) hold for any $u \in X$ and $\nu \in \mathcal{P}(X)$.

Remark 2.2. Let us underline that in condition (ii), the initial point u belongs to X and the final one $\hat{u}$ to $\mathcal{A}$, so the semigroup is not irreducible in X. Also note that (iii) and (iv) are necessary conditions for convergence (2.3). Indeed, (2.3) is equivalent to

$$\sup_{\|f\|_L \leq 1} \|\lambda^{-k}\mathfrak{P}_k f - \langle f, \mu\rangle h\|_X \leq Ce^{-\gamma k}.$$

By taking any non-negative function $f \in L_b(X)$ that vanishes on $\mathcal{A}$ and equals 1 outside $\mathcal{A}_r$, we get (iii), and taking $f = \mathbf{1}$, we get (iv).

Furthermore, in the case $X = \mathcal{A}$, (iii) is trivially satisfied and (iv) follows from (i) and (ii). Indeed, if conditions (i) and (ii) hold, we can apply Theorem 2.1 in [14] on the set $\mathcal{A}$:

$$\|\lambda^{-k}\mathfrak{P}_k f - \langle f, \mu\rangle h\|_{\mathcal{A}} \to 0 \quad \text{as } k \to \infty \text{ for } f \in L_b(\mathcal{A}). \tag{2.7}$$

Choosing $f = \mathbf{1}$, we get (iv).

We conclude from this remark that it suffices to prove only the second assertion of Theorem 2.1. Its proof is divided into three parts.

2.1 Existence of eigenvectors μ and h

The existence of an eigenvector $\mu \in \mathcal{P}(\mathcal{A})$ is shown in [14], by applying the Leray–Schauder theorem to the continuous mapping

$$F : \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{A}), \quad F(\mu) = (\mathfrak{P}_1^*\mu(\mathcal{A}))^{-1}\mathfrak{P}_1^*\mu.$$

Any fixed point μ of F is an eigenvector for $\mathfrak{P}_1^*$ corresponding to an eigenvalue $\lambda = \mathfrak{P}^*\mu(\mathcal{A}) > 0$. The irreducibility on $\mathcal{A}$ implies that $\operatorname{supp}\mu = \mathcal{A}$. Note that replacing $P(u,\Gamma)$ by $\lambda^{-1}P(u,\Gamma)$, we may assume that $\lambda = 1$. From now on, we shall always assume that $\lambda = 1$, without further stipulation.

Let us show the existence of a Lipschitz-continuous function $h : X \to \mathbb{R}_+$ satisfying (2.2) for any $u \in X$. We use some arguments from [14].

Step 1: Existence. Using (2.1) and (2.6), we see that the sequence $\{\mathfrak{P}_k \mathbf{1}\}$ is uniformly equicontinuous on X. It follows that so is the sequence

$$h_k = \frac{1}{k}\sum_{n=1}^{k} \mathfrak{P}_n \mathbf{1}. \tag{2.8}$$

Applying the Arzelà–Ascoli theorem, we construct a function $h : X \to \mathbb{R}_+$ and a sequence $k_j \to \infty$ such that

$$\|h_{k_j} - h\|_X \to 0 \quad \text{as } j \to \infty. \tag{2.9}$$

Passing to the limit as $j \to \infty$ in the equality

$$\mathfrak{P}_1 h_{k_j}(u) = h_{k_j}(u) + \frac{1}{k_j}\big(\mathfrak{P}_{k_j+1}\mathbf{1}(u) - \mathfrak{P}_1\mathbf{1}(u)\big), \quad u \in X,$$

and using (2.6), we get (2.2). The Lipschitz-continuity of h follows from (2.1), (2.6), (2.8), and (2.9).

Step 2: Positivity. Now we show that $h(u) > 0$ for any $u \in X$. Using equalities $\mathfrak{P}_k^*\mu = \mu$ and (2.8) together with limit (2.9), we obtain

$$1 = \langle h_{k_j}, \mu\rangle \to \langle h, \mu\rangle \quad \text{as } j \to \infty.$$

So $\langle h, \mu\rangle = 1$ and $h(\hat{u}) > 0$ for some $\hat{u} \in \mathcal{A}$. By the continuity of h, there is $r > 0$ such that $h(v) \geq r$ for any $v \in B_r(\hat{u})$. Thanks to (ii), for sufficiently large $m \geq 1$,

$$h(u) = \mathfrak{P}_m h(u) \geq r\, P_m\big(u, B_r(\hat{u})\big) > 0 \qquad \text{for any } u \in X.$$

This completes the proof of existence of eigenvectors μ and h. The uniqueness will follow from inequality (2.3).

2.2 A weak version of (2.3)

In this section, we show that the left-hand side of (2.3) converges to zero. We shall use this in the next section to establish exponential convergence.

Proposition 2.3. *Under Conditions* (i)-(iv)*, we have*

$$\sup_{\nu \in \mathcal{P}(X)} \|\mathfrak{P}_k^*\nu - \langle h, \nu\rangle\mu\|_L^* \to 0 \qquad \text{as } k \to \infty. \tag{2.10}$$

Proof. It suffices to prove the limit

$$\sup_{\|f\|_L \leq 1,\, \langle f, \mu\rangle = 0} \|\mathfrak{P}_k f\|_X \to 0 \qquad \text{as } k \to \infty. \tag{2.11}$$

Let us represent $k = l + m$, use the semigroup property, and write for any $r > 0$,

$$\mathfrak{P}_k f(u) = \mathfrak{P}_l\left(\mathbb{1}_{\mathcal{A}_r}\mathfrak{P}_m f\right)(u) + \mathfrak{P}_l\left(\mathbb{1}_{X\setminus\mathcal{A}_r}\mathfrak{P}_m f\right)(u) = \mathcal{I}_1 + \mathcal{I}_2, \tag{2.12}$$

where $\mathcal{A}_r$ is defined by (2.5).

Step 1: Estimate for $\mathcal{I}_1$. Inequalities (2.1), (2.6), and $\|f\|_L \leq 1$ imply that

$$|\mathfrak{P}_m f(v) - \mathfrak{P}_m f(v')| \leq C_1 \Lambda\, d(v, v'), \quad m \geq 1, v, v' \in X$$

for some number $C_1 > 0$. Combining this with the definition of $\mathcal{A}_r$, we see that

$$|\mathcal{I}_1| \leq C_1 \Lambda\, r\, \mathfrak{P}_l\mathbf{1}(u) + \|\mathfrak{P}_m f\|_{\mathcal{A}}\, \mathfrak{P}_l\mathbf{1}(u).$$

Taking the supremum over $u \in X$ and using (2.6), we get

$$\|\mathcal{I}_1\|_X \le C_1 \Lambda^2 r + \|\mathfrak{P}_m f\|_{\mathcal{A}} \Lambda.$$

Moreover, by virtue of (2.7),

$$\sup_{\|f\|_L \le 1,\, \langle f, \mu \rangle = 0} \|\mathfrak{P}_m f\|_{\mathcal{A}} \to 0 \qquad \text{as } m \to \infty,$$

whence

$$\sup_{\|f\|_L \le 1,\, \langle f, \mu \rangle = 0} \|\mathcal{I}_1\|_X \le C_2 r \tag{2.13}$$

for arbitrary $r > 0$ and $l \ge 1$, and sufficiently large $m = m(r) \ge 1$.

Step 2: Estimate for $\mathcal{I}_2$. Let us fix m such that (2.13) holds. Using the inequalities $\|f\|_X \le 1$ and (2.6) together with (2.4), we obtain

$$\mathcal{I}_2 \le \Lambda \, \|P_l(\cdot, X \setminus \mathcal{A}_r)\|_X \to 0 \qquad \text{as } l \to \infty.$$

Combining this with (2.12) and (2.13), we get

$$\|\mathfrak{P}_k f\|_X \le \|\mathcal{I}_1\|_X + \|\mathcal{I}_2\|_X \le 2\, C_2 r$$

for $k = k(r) \ge 1$ sufficiently large. Since $r > 0$ can be chosen arbitrarily small, we arrive at (2.11).

□

2.3 The rate of convergence

In this section, we show that the rate of convergence in (2.10) is exponential. To this end, we introduce an auxiliary Markov semigroup $\mathscr{S}_k$ acting on $C(X)$ by

$$\mathscr{S}_k g = h^{-1} \mathfrak{P}_k(gh) \qquad \text{for } g \in C(X).$$

The following result shows that $\mathscr{S}_k$ is exponentially mixing.

Proposition 2.4. *Under the conditions of Theorem 2.1, the semigroup $\mathscr{S}_k$ has a unique stationary measure given by $\hat{\sigma} = h\mu$. Moreover, we have*

$$\|\mathscr{S}_k^* \sigma - \hat{\sigma}\|_L^* \le C e^{-\gamma k} \qquad \text{for } \sigma \in \mathcal{P}(X),\ k \ge 1, \tag{2.14}$$

where $C > 0$ and $\gamma > 0$ are some numbers not depending on σ and k.

Taking this proposition for granted, let us prove (2.3). Choosing $\sigma = \delta_u$ in (2.14), where δ_u is the Dirac measure concentrated at $u \in X$, we see that

$$|\mathfrak{P}_k(gh)(u) - \langle gh, \mu \rangle h(u)| \le C e^{-\gamma k} h(u) \le C_1 e^{-\gamma k}, \qquad k \ge 1$$

for any $g \in L_b(X)$ with $\|g\|_L \le 1$. Since h is positive and Lipschitz, any $f \in L_b(X)$ can be represented as $f = gh$ for some $g \in L_b(X)$, which leads to (2.3).

The rest of the section is devoted to the proof of Proposition 2.4. Note that the equality

$$\mathscr{S}_1^* \sigma = h\mathfrak{P}_1^*(h^{-1}\sigma)$$

and the fact that μ is an eigenvector for $\mathfrak{P}_1^*$ imply that $\hat{\sigma} = h\mu$ is a stationary measure for $\mathscr{S}_1^*$. To prove the uniqueness and exponential mixing, we will show that the operator $\mathscr{S}_m^*$ is a contraction if the space X is endowed with an appropriate metric and $m \geq 1$ is sufficiently large. The proof relies on the refined uniform Feller property.

Let us endow X with the metric d_θ given by

$$d_\theta(u, v) = 1 \wedge (\theta\, d(u, v)),$$

where $\theta > 0$ is a large number that will be fixed later. We consider the Kantorovich metric on $\mathcal{P}(X)$ defined by

$$\|\sigma_1 - \sigma_2\|_{K_\theta} = \sup_{L_\theta(f)\leq 1} |\langle f, \sigma_1\rangle - \langle f, \sigma_2\rangle|, \quad \sigma_1, \sigma_2 \in \mathcal{P}(X),$$

where

$$L_\theta(f) = \sup_{u,v\in X, u\neq v} \frac{|f(u) - f(v)|}{d_\theta(u, v)}.$$

Proposition 2.4 follows immediately from the following two lemmas.

Lemma 2.5. *For any $\theta \geq (\mathrm{diam}(X))^{-1}$ and $\sigma_1, \sigma_2 \in \mathcal{M}_+(X)$, we have*

$$\frac{1}{1+\theta}\|\sigma_1 - \sigma_2\|_{K_\theta} \leq \|\sigma_1 - \sigma_2\|_L^* \leq \mathrm{diam}(X)\|\sigma_1 - \sigma_2\|_{K_\theta}, \tag{2.15}$$

where $\mathrm{diam}(X) = \sup_{u,v\in X} d(u, v)$.

Lemma 2.6. *For sufficiently large number $\theta > 0$ and integer $m \geq 1$, we have*

$$\|\mathscr{S}_m^*\sigma_1 - \mathscr{S}_m^*\sigma_2\|_{K_\theta} \leq \frac{1}{2}\|\sigma_1 - \sigma_2\|_{K_\theta} \qquad \text{for } \sigma_1, \sigma_2 \in \mathcal{P}(X). \tag{2.16}$$

Proof of Lemma 2.5. Step 1. Let us prove the first inequality in (2.15). We take any $f \in L_b(X)$ with $L_\theta(f) \leq 1$. Replacing f by $f - f(0)$, we may assume that $f(0) = 0$. Then, the definition of the metric d_θ implies that

$$\|f\|_X \leq 1 \quad \text{and} \quad L(f) \leq \theta, \tag{2.17}$$

where

$$L(f) = \sup_{u,v\in X, u\neq v} \frac{|f(u) - f(v)|}{d(u, v)}.$$

Thus $\|f\|_L \leq 1 + \theta$ and we obtain the required inequality.

Step 2. Let us take any $f \in L_b(X)$ with $\|f\|_L \leq 1$. The second inequality in (2.15) will be proved, if we show that $L_\theta(f) \leq \mathrm{diam}(X)$. We claim that this is the case for $\theta \geq (\mathrm{diam}(X))^{-1}$. Indeed, since $L(f) \leq \|f\|_L$, we have

$$|f(u) - f(v)| \leq d(u, v) \leq \mathrm{diam}(X) \wedge (\mathrm{diam}(X)\theta d(u, v)) \leq \mathrm{diam}(X) d_\theta(u, v)$$

for any $u, v \in X$. This completes the proof of Lemma 2.5. □

Proof of Lemma 2.6. Step 1. Inequality (2.16) will be established if we show that for any $f \in L_b(X)$ with $L_\theta(f) \le 1$, there is a function $g \in L_b(X)$ with $L_\theta(g) \le 1$ such that

$$|\langle \mathscr{S}_m f, \sigma_1 \rangle - \langle \mathscr{S}_m f, \sigma_2 \rangle| \le \frac{1}{2} |\langle g, \sigma_1 \rangle - \langle g, \sigma_2 \rangle|, \quad \sigma_1, \sigma_2 \in \mathcal{P}(X)$$

for some constant $\theta > 0$ and integer $m \ge 1$ not depending on f. As above, we may assume that f vanishes at the origin. Note that the above inequality is trivially satisfied with $g = 2\mathscr{S}_m f$. Therefore, we only need to show that for an appropriate choice of θ and m we have $L_\theta(g) \le 1$ or equivalently

$$L_\theta(\mathscr{S}_m f) \le \frac{1}{2}. \tag{2.18}$$

Step 2. By virtue of (2.1), there is a positive constant C such that

$$L(\mathscr{S}_m f) \le C \|f\|_\infty + \frac{1}{4} L(f), \quad m \ge 1.$$

Using this inequality together with (2.17), we get

$$L(\mathscr{S}_m f) \le C + \frac{\theta}{4} \le \frac{\theta}{2}, \quad m \ge 1, \tag{2.19}$$

if $\theta \ge 4C$. Further, thanks to (2.10), we have

$$\sup_{u,v \in X} \|\mathscr{S}_k^* \delta_u - \mathscr{S}_k^* \delta_v\|_L^* \to 0 \qquad \text{as } k \to \infty.$$

Note that $\|f\|_L \le 1 + \theta$. Therefore, we can find an integer $m = m(\theta)$ such that

$$\sup_{u,v \in X} |\mathscr{S}_m f(u) - \mathscr{S}_m f(v)| \le \frac{1}{2}.$$

Combining this with (2.19), we arrive at (2.18) □

3 Proof of Theorems 1.1-1.3

Before starting the proofs, let us make a few comments about the assumptions entering these theorems. Hypothesis (D) ensures that the support $\mathcal{K}$ of the law of η_1 is contained in a Hilbert cube, so it is a compact set in H. Since S is locally Lipschitz in H, we have

$$\|S(u_1) - S(u_2)\| \le C_R \|u_1 - u_2\|, \quad u_1, u_2 \in B_R.$$

Combining this with (1.3), we infer that the mapping $S : H \to H$ is compact, i.e., the image under S of any bounded set is relatively compact. Thus the domain of attainability $\mathcal{A}(B)$ from any bounded set $B \subset H$ is compact. By definition, $\mathcal{A}(B)$ is invariant for (0.1). Therefore $\mathcal{A}(B_\rho)$ is a compact invariant absorbing set, where ρ is the number in (1.2).

Let us give here the details of the proof of Theorem 1.1. We apply Theorem 2.1 for the generalised Markov kernel

$$P_1^V(u,\Gamma) = \mathbb{E}_u \left\{ \mathbb{1}_{\{u_1\in\Gamma\}} e^{V(u_1)} \right\} = \int_\Gamma P_1(u,\mathrm{d}z)\, e^{V(z)},$$

where $P_1(u,\Gamma)$ is the transition function of $(u_k, \mathbb{P}_u)$, $u \in \mathcal{A}$, $\Gamma \in \mathcal{B}(\mathcal{A})$, and $V \in L_b(\mathcal{A})$. Then Condition (i) follows from Proposition 4.1 applied for $B = \{0\}$, and (ii) is proved in the following lemma. Applying the first assertion of Theorem 2.1, we complete the proof of Theorem 1.1.

Lemma 3.1. *Under Conditions* (A)-(D)*, for any $V \in L_b(H)$, $R > 0$, and $r > 0$, there is an integer $m \geq 1$ and a number $p > 0$ such that*

$$P_m^V(u, B_r(\hat{u})) \geq p \quad \textit{for all } u \in \mathcal{A}(B_R) \textit{ and } \hat{u} \in \mathcal{A}.$$

Proof. As V is bounded, we have

$$P_m^V(u,\Gamma) \geq e^{-m\|V\|_\infty} P_m(u,\Gamma), \qquad \text{for } u \in \mathcal{A}(B_R),\, \Gamma \in \mathcal{B}(X).$$

Using (1.1), the inclusion $0 \in \mathcal{K}$ (see Condition (D)), and a simple compactness argument, we can choose the numbers $r, p > 0$ and the integer $m \geq 1$ such that

$$P_m(u, B_r(\hat{u})) \geq p \quad \text{ for } u \in X,\, \hat{u} \in \mathcal{A}.$$

This implies the required result. ☐

We establish Theorems 1.2 and 1.3 in the following two sections.

3.1 The case of a potential with a small oscillation

Theorem 1.2 is proved by applying the second assertion of Theorem 2.1 for the kernel P_1^V in the compact space $X = \mathcal{A}(B_\rho)$. Since $\mathcal{A}$ is an invariant set for (0.1), we have $P_1^V(u, X\setminus\mathcal{A}) = 0$ for $u \in \mathcal{A}$. Conditions (i) and (ii) are established in Proposition 4.1 and Lemma 3.1. Thus Theorem 1.2 will be proved if we check (iii) and (iv). Indeed, by Theorem 2.1, we will then have inequality (1.5) for any $u \in \mathcal{A}(B_\rho)$, hence also for any $u \in B_R$ by the absorbing property (B).

We shall prove (iii) and (iv) for a potential V with a sufficiently small oscillation. Without loss of generality, we can always assume that $\lambda_V = 1$. Indeed, it suffices to replace V by $V - \log\lambda_V$ (this has no impact on the oscillation of V).

3.1.1 Condition (iii)

Lemma 3.2. *Under Conditions* (A)-(D)*, there is a number $\delta > 0$ such that for any $V \in L_b(H)$ with* $\mathrm{Osc}(V) \le \delta$ *and any $r > 0$, we have*

$$\|P_k^V(\cdot, X \setminus \mathcal{A}_r)\|_X \to 0 \qquad \text{as } k \to \infty. \tag{3.1}$$

Proof. Let us show that if $\mathrm{Osc}(V)$ is sufficiently small, then

$$\sup_{u \in X} \mathbb{E}_u \left\{ \mathbb{1}_{\{u_k \notin \mathcal{A}_r\}} \Xi_k^V \mathbf{1} \right\} \le C(r, \rho) e^{-\alpha k/2}, \quad k \ge 1, \tag{3.2}$$

where $\alpha > 0$ is the number in (0.2) and Ξ_k^V is defined by (0.3). Indeed, let $f \in L_b(H)$ be a non-negative function that vanishes on $\mathcal{A}$ and equals 1 outside $\mathcal{A}_r$. Observe that, since $\mathcal{A}$ is invariant and compact, it contains[4] the support of the unique stationary measure $\mu \in \mathcal{P}(H)$ of $(u_k, \mathbb{P}_u)$. Thus $\langle f, \mu \rangle = 0$. Note that

$$\text{as } \lambda_V = 1, \text{ we have } \inf_{u \in \mathcal{A}} V(u) \le 0 \text{ and } \|V\|_\infty \le \mathrm{Osc}(V) \le \delta. \tag{3.3}$$

Combining this with (0.2), we obtain

$$\mathbb{E}_u \big\{ \mathbb{1}_{\{u_k \notin \mathcal{A}_r\}} \Xi_k^V \mathbf{1} \big\} \le e^{k\|V\|_\infty} \mathbb{E}_u f(u_k) \le C(r, \rho) e^{k \,\mathrm{Osc}(V)} e^{-\alpha k}.$$

Now assuming $\delta \le \alpha/2$, we arrive at (3.2). □

3.1.2 Condition (iv)

Lemma 3.3. *Under Conditions* (A)-(D)*, there is a number $\delta > 0$ such that for any $V \in L_b(H)$ with* $\mathrm{Osc}(V) \le \delta$*,*

$$\Lambda_R = \sup_{k \ge 1} \|\mathfrak{P}_k^V \mathbf{1}\|_{\mathcal{A}(B_R)} < \infty \qquad \text{for any } R > 0. \tag{3.4}$$

Proof. Step 1. It suffices to prove the lemma for $R = \rho$. For any $\varepsilon > 0$, let τ_ε be the first hitting time of the ball B_ε:

$$\tau_\varepsilon = \min\{k \ge 0 : u_k \in B_\varepsilon\}.$$

Let us show that for some $\delta > 0$, we have

$$\sup_{u \in X} \mathbb{E}_u e^{\delta \tau_\varepsilon} \le 2. \tag{3.5}$$

Indeed, by Conditions (A) and (D), there are $q \in (0, 1)$ and $l \ge 1$ such that

$$\mathbb{P}_u\{u_l \in B_\varepsilon\} \ge q, \quad u \in X.$$

Then for any $k \ge 1$, the Markov property gives

$$\begin{aligned} \mathbb{P}_u\{kl < \tau_\varepsilon\} &\le \mathbb{P}_u\{u_{jl} \notin B_\varepsilon, \, j = 0, \dots, k\} \\ &\le (1-q)\mathbb{P}_u\{u_{jl} \notin B_\varepsilon, \, j = 0, \dots, k-1\} \le (1-q)^k, \quad u \in X, \end{aligned}$$

[4]The irreducibility property on $\mathcal{A}$ implies that $\mathcal{A} = \operatorname{supp} \mu$.

which allows to conclude that $\sup_{u\in X} \mathbb{E}_u e^{\delta\tau_\varepsilon}$ is finite for some $\delta > 0$. Choosing a smaller δ and using the Hölder inequality, we obtain (3.5).

Step 2. Using (3.3) and the strong Markov property, we get for $u \in X$,

$$\mathfrak{P}_k^V \mathbf{1}(u) = \mathbb{E}_u \left\{ \mathbb{1}_{\{\tau \geq k\}} \Xi_k^V \mathbf{1} \right\} + \mathbb{E}_u \left\{ \mathbb{1}_{\{\tau < k\}} \Xi_k^V \mathbf{1} \right\} \leq \mathbb{E}_u e^{\delta\tau_\varepsilon} + \mathbb{E}_u \left\{ e^{\delta\tau_\varepsilon} \mathfrak{P}_k^V \mathbf{1}(u_{\tau_\varepsilon}) \right\}.$$

From (3.5) we derive

$$\|\mathfrak{P}_k^V \mathbf{1}\|_X \leq 2 + 2\|\mathfrak{P}_k^V \mathbf{1}\|_{B_\varepsilon \cap X}. \tag{3.6}$$

On the other hand, using inequality

$$|\mathfrak{P}_k^V f(v) - \mathfrak{P}_k^V f(v')| \leq C\|f\|_L \, \|\mathfrak{P}_k^V \mathbf{1}\|_{\mathcal{A}(B)} \|v - v'\|$$

that follows from Proposition 4.1, with $B = B_\rho$ and $f = \mathbf{1}$, we get

$$\left|\mathfrak{P}_k^V \mathbf{1}(u) - \mathfrak{P}_k^V \mathbf{1}(0)\right| \leq \frac{1}{4} \|\mathfrak{P}_k^V \mathbf{1}\|_X, \quad u \in B_\varepsilon \cap X, \, k \geq 1$$

for sufficiently small $\varepsilon > 0$. Combining this with (3.6), we infer

$$\|\mathfrak{P}_k^V \mathbf{1}\|_X \leq 4 + 4\,\mathfrak{P}_k^V \mathbf{1}(0) \leq 4 + 4\|\mathfrak{P}_k^V \mathbf{1}\|_{\mathcal{A}}.$$

To conclude, it remains to recall that the sequence $\{\|\mathfrak{P}_k^V \mathbf{1}\|_{\mathcal{A}}\}$ is bounded, by virtue of (2.7). $\square$

3.2 The case of an arbitrary potential

As in the previous section, to prove Theorem 1.3, we need to check (iii) and (iv) in the space $X = \mathcal{A}(B_\rho)$. The arguments are more involved, since the oscillation of V now can be arbitrarily large.

3.2.1 Condition (iii)

Lemma 3.4. *Under Conditions* (A)-(E)*, for any* $V \in L_b(H)$ *and* $r > 0$*, we have limit* (3.1)*.*

Proof. By Remark 3.7, there is $\varepsilon \in (0, r)$ such that the inclusion $\mathcal{A}(\mathcal{A}_\varepsilon) \subset \mathcal{A}_r$ holds. Note that this implies the following: if for some $k \geq 1$ and $\omega \in \Omega$, $u_k(\omega) \notin \mathcal{A}_r$, then $u_0(\omega) \notin \mathcal{A}_\varepsilon, \ldots, u_{k-1}(\omega) \notin \mathcal{A}_\varepsilon$, and since $\varepsilon < r$, we also have $u_k(\omega) \notin \mathcal{A}_\varepsilon$. Let V_ε be a Lipschitz-continuous function that vanishes on $\mathcal{A}$ and coincides with V outside $\mathcal{A}_\varepsilon$. It follows that

$$\mathbb{1}_{\{u_k \notin \mathcal{A}_r\}} \Xi_k^V \mathbf{1} = \mathbb{1}_{\{u_k \notin \mathcal{A}_r\}} \Xi_k^{V_\varepsilon} \mathbf{1}.$$

Taking the expectation and using the Cauchy–Schwartz inequality, we get

$$\mathbb{E}_u \left\{ \mathbb{1}_{\{u_k \notin \mathcal{A}_r\}} \Xi_k^V \mathbf{1} \right\} = \mathbb{E}_u \left\{ \mathbb{1}_{\{u_k \notin \mathcal{A}_r\}} \Xi_k^{V_\varepsilon} \mathbf{1} \right\} \leq \left(\mathbb{E}_u \mathbb{1}_{\{u_k \notin \mathcal{A}_r\}} \right)^{1/2} \left(\mathfrak{P}_k^{\mathbb{V}} \mathbf{1}(u) \right)^{1/2},$$

where we set $\mathbb{V} = 2V_\varepsilon$. Further, since the function V_ε vanishes on $\mathcal{A}$, so does $\mathbb{V}$ and hence $\lambda_{\mathbb{V}} = 1$. In view of Lemma 3.5,

$$\|\mathfrak{P}_k^{\mathbb{V}}\mathbf{1}\|_X \le \Lambda_\rho, \quad k \ge 1.$$

Let f be a non-negative Lipschitz-continuous function vanishing on $\mathcal{A}$ and 1 outside $\mathcal{A}_r$. Then, due to exponential mixing (0.2), we have

$$\sup_{u\in X} \mathbb{E}_u \mathbb{1}_{\{u_k \notin \mathcal{A}_r\}} \le \sup_{u\in X} \mathbb{E}_u f(u_k) \le C(r,\rho)e^{-\alpha k}, \quad k \ge 1.$$

Combining last three inequalities, we arrive at (3.1).

□

3.2.2 Condition (iv)

Lemma 3.5. *Under Conditions* (A)-(E)*, for any* $V \in L_b(H)$ *and* $R > 0$*, we have inequality* (3.4)*.*

Proof. We shall use a bootstrap argument to establish this result. Let

$$R_* = \sup\{R \ge 0 : \Lambda_R < \infty\}.$$

The lemma will be proved if we show that $R_* = \infty$.

Step 1. We first show that if Λ_R is finite for some $R \ge 0$, then so is $\Lambda_{R+\varepsilon}$ for some $\varepsilon \in (0,1)$. To this end, first note that in view of inequality (4.1), we have

$$|\mathfrak{P}_k^V f(v) - \mathfrak{P}_k^V f(v')| \le C_R \,\|f\|_L \|\mathfrak{P}_k^V \mathbf{1}\|_{\mathcal{A}(B)} \|v - v'\| \tag{3.7}$$

for any $B \subset B_{R+1}$ and $v, v' \in \mathcal{A}(B)$. Applying inequality (3.7) with $f = \mathbf{1}$ and $B = B_{R+\varepsilon}$, we get

$$|\mathfrak{P}_k^V \mathbf{1}(u)| \le |\mathfrak{P}_k^V \mathbf{1}(v)| + C_R \|\mathfrak{P}_k^V \mathbf{1}\|_{\mathcal{A}(B_{R+\varepsilon})} \|u - v\|, \quad u, v \in \mathcal{A}(B_{R+\varepsilon}).$$

In particular, this inequality is true for any $v \in \mathcal{A}(B_R)$ and $u \in \mathcal{A}(B_{R+\varepsilon})$. Therefore, taking first the infimum over $v \in \mathcal{A}(B_R)$ and then supremum over $u \in \mathcal{A}(B_{R+\varepsilon})$, we derive

$$\begin{aligned}\|\mathfrak{P}_k^V \mathbf{1}\|_{\mathcal{A}(B_{R+\varepsilon})} &\le \Lambda_R + C_R \|\mathfrak{P}_k^V \mathbf{1}\|_{\mathcal{A}(B_{R+\varepsilon})} \sup_{u\in\mathcal{A}(B_{R+\varepsilon})} \inf_{v\in\mathcal{A}(B_R)} \|u - v\| \\ &= \Lambda_R + C_R \|\mathfrak{P}_k^V \mathbf{1}\|_{\mathcal{A}(B_{R+\varepsilon})}\, d_H(\mathcal{A}(B_{R+\varepsilon}), \mathcal{A}(B_R)),\end{aligned} \tag{3.8}$$

where $d_H(E, F)$ is the Hausdorff distance between the sets $E, F \subset H$. We use the following result proved in the next section.

Lemma 3.6. *For any* $R \ge 0$*, we have* $d_H(\mathcal{A}(B_{R+\varepsilon}), \mathcal{A}(B_R)) \to 0$ *as* $\varepsilon \downarrow 0$*. Moreover, if* $R > 0$*, we have also* $d_H(\mathcal{A}(B_{R-\varepsilon}), \mathcal{A}(B_R)) \to 0$ *as* $\varepsilon \downarrow 0$*.*

In view of the first assertion of this lemma,

$$d_H(\mathcal{A}(B_{R+\varepsilon}), \mathcal{A}(B_R)) \leq \frac{1}{2C_R}$$

for $\varepsilon > 0$ sufficiently small. Combining this with (3.8), we get

$$\|\mathfrak{P}_k^V \mathbf{1}\|_{\mathcal{A}(B_{R+\varepsilon})} \leq 2\,\Lambda_R, \quad k \geq 1,$$

which implies $\Lambda_{R+\varepsilon} \leq 2\,\Lambda_R < \infty$.

Step 2. In this step, we show that $R_* = \infty$. Note that for $R = 0$, we have $\mathcal{A}(B_R) = \mathcal{A}$, so that Λ_0 is finite in view of (2.7). The result of the previous step implies that $R_* > 0$ and if $R_* < \infty$, then it cannot be attained, i.e., $\Lambda_{R_*} = \infty$. In search of a contradiction, assume that $R_* < \infty$ and take any $\varepsilon \in (0, R_*)$. As above, we apply inequality (3.7) with $f = \mathbf{1}$ and $B = B_{R_*}$:

$$|\mathfrak{P}_k^V \mathbf{1}(u)| \leq |\mathfrak{P}_k^V \mathbf{1}(v)| + C_{R_*} \|\mathfrak{P}_k^V \mathbf{1}\|_{\mathcal{A}(B_{R_*})} \|u - v\|, \quad u, v \in \mathcal{A}(B_{R_*}).$$

Taking first the infimum over $v \in \mathcal{A}(B_{R_*-\varepsilon})$ and then the supremum over $u \in \mathcal{A}(B_{R_*})$, we obtain

$$\|\mathfrak{P}_k^V \mathbf{1}\|_{\mathcal{A}(B_{R_*})} \leq \Lambda_{R_*-\varepsilon} + C_{R_*} \|\mathfrak{P}_k^V \mathbf{1}\|_{\mathcal{A}(B_{R_*})} d_H(\mathcal{A}(B_{R_*}), \mathcal{A}(B_{R_*-\varepsilon})).$$

Using the second assertion of Lemma 3.6, for sufficiently small $\varepsilon > 0$, we get

$$d_H(\mathcal{A}(B_{R_*}), \mathcal{A}(B_{R_*-\varepsilon})) \leq \frac{1}{2C_{R_*}}.$$

We thus infer

$$\|\mathfrak{P}_k^V \mathbf{1}\|_{\mathcal{A}(B_{R_*})} \leq 2\,\Lambda_{R_*-\varepsilon}, \quad k \geq 1.$$

This contradiction proves that $R_* = \infty$. ☐

3.2.3 Proof of Lemma 3.6

The second part of the lemma readily follows from the definition of $\mathcal{A}(B_R)$ and its compactness. The first one is more delicate, and this is where we use Condition (E). Clearly, it is sufficient to show that

$$\mathcal{A}(B_R) = \bigcap_{\varepsilon > 0} \mathcal{A}(B_{R+\varepsilon}). \tag{3.9}$$

The proof of this equality is divided into two steps.

Step 1: Reduction. Without loss of generality, we can assume that R is smaller than the number ρ in (B). For any $r > 0$, we introduce the set

$$\mathcal{C}_r = \{u \in Y : d'(u, \mathcal{A}(B_R)) \leq r\},$$

where d' is the metric in (E) and $Y = \mathcal{A}(B_{\rho+1})$. We claim that (3.9) will be established if we show that for any $r > 0$ there is $\varepsilon > 0$ such that

$$\mathcal{A}(m, B_{R+\varepsilon}) \subset \mathcal{C}_r \quad \text{for any } m \geq 1. \tag{3.10}$$

Indeed, once this is proved, we will have

$$\bigcup_{m=1}^{\infty} \mathcal{A}(m, B_{R+\varepsilon}) \subset \mathcal{C}_r. \tag{3.11}$$

Now note that the set $\mathcal{C}_r$ is closed in H with respect to the natural topology. Indeed, if the sequence $\{u_k\} \subset \mathcal{C}_r$ converges to u in H, then applying the triangle inequality, we obtain

$$d'(u, \mathcal{A}(B_R)) \le d'(u, u_k) + d'(u_k, \mathcal{A}(B_R)) \le d'(u, u_k) + r.$$

Letting k go to infinity and using the fact that the convergence in H implies the one in d', we get $u \in \mathcal{C}_r$. Therefore, taking the closure in H in the inclusion (3.11), we see that $\mathcal{A}(B_{R+\varepsilon}) \subset \mathcal{C}_r$. Letting r go to zero, we arrive at (3.9).

Step 2: Derivation of (3.10). Let us fix any $r > 0$. First note that, since the topology of d' is weaker than the natural one of H, for any $u \in Y$, there is $a > 0$ such that $d'(u, v) \le r$, provided $\|u - v\| \le a$. Using the compactness of Y, we see that a can be taken uniformly for $u \in Y$. Let us show that (3.10) holds for sufficiently small $\varepsilon > 0$. Indeed, take any $m \ge 1$ and $u_* \in \mathcal{A}(m, B_{R+\varepsilon})$. Clearly, $u_* \in Y$ if $R + \varepsilon \le \rho + 1$. To show that $d'(u_*, \mathcal{A}(B_R)) \le r$, note that there are $u_0 \in B_{R+\varepsilon}$ and $\eta_1, \ldots, \eta_m \in \mathcal{K}$ verifying

$$u_1 = S(u_0) + \eta_1, \ldots, u_m = S(u_{m-1}) + \eta_m$$

with $u_* = u_m$. Let us take any $v_0 \in B_R$ such that $\|u_0 - v_0\| \le \varepsilon$ and define

$$v_1 = S(v_0) + \eta_1, \ldots, v_m = S(v_{m-1}) + \eta_m.$$

Using the translation invariance of d' and inequality (1.6), we obtain

$$d'(u_m, v_m) = d'(S(u_{m-1}), S(v_{m-1})) \le d'(u_{m-1}, v_{m-1}).$$

Iterating this and using $u_* = u_m$, we arrive at

$$d'(u_*, v_m) \le d'(u_1, v_1).$$

But for sufficiently small ε we have

$$\|u_1 - v_1\| = \|S(u_0) - S(v_0)\| \le a.$$

Thus $d'(u_1, v_1) \le r$, by definition of a. This implies that $d'(u_*, v_m) \le r$, and to conclude, it remains to note that v_i all belong to $\mathcal{A}(B_R)$ by definition.

Remark 3.7. Literally repeating the argument of the proof of (3.9), we get

$$\bigcap_{\varepsilon > 0} \mathcal{A}(\mathcal{A}_\varepsilon) = \mathcal{A}.$$

This implies that for any $r > 0$, there is $\varepsilon > 0$ such that $\mathcal{A}(\mathcal{A}_\varepsilon) \subset \mathcal{A}_r$.

4 Refined uniform Feller property

This section is devoted to the proof of the following result.

Proposition 4.1. *Under Conditions* (A)-(D), *for any* $V \in L_b(H)$, $R > 0$, *and* $c \in (0,1)$, *there is a number* $C = C(\|V\|_L, R, c) > 0$ *such that*

$$|\mathfrak{P}_k^V f(v) - \mathfrak{P}_k^V f(v')| \le \big(C\,\|f\|_\infty + c^k\,\|f\|_L\big)\,\|\mathfrak{P}_k^V \mathbf{1}\|_{\mathcal{A}(B)}\|v - v'\| \tag{4.1}$$

for any set $B \subset B_R$, *initial points* $v, v' \in \mathcal{A}(B)$, *and function* $f \in L_b(H)$.

We prove this proposition by developing the ideas of the proof of the uniform Feller property on $\mathcal{A}$ established in Theorem 3.1 in [14]. We start by recalling the properties of the coupling process. Let $\mathbf{P}(v)$ be the law of the trajectory $\{u_k\}$ for (0.1) issued from $v \in \mathcal{A}(B)$, i.e., $\mathbf{P}(v)$ is a probability measure on the direct product of countably many copies of $\mathcal{A}(B)$. The following result is a version of Proposition 3.2 in [14]; see Section 3.2.2 in [21] for the proof.

Proposition 4.2. *For sufficiently large integer* $N \ge 1$ *there is a probability space* $(\Omega_N, \mathcal{F}_N, \mathbb{P}_N)$ *and an* $\mathcal{A}(B)\times\mathcal{A}(B)$*-valued Markov process* (u_k, u_k') *on* Ω_N *parametrised by the initial point* $(v, v') \in \mathcal{A}(B) \times \mathcal{A}(B)$ *for which the following properties hold.*

- **(a)** *The* $\mathbb{P}_N$*-laws of the sequences* $\{u_k\}$ *and* $\{u_k'\}$ *coincide with* $\mathbf{P}(v)$ *and* $\mathbf{P}(v')$, *respectively.*
- **(b)** *The projections* $\mathsf{Q}_N(u_k - S(u_{k-1}))$ *and* $\mathsf{Q}_N(u_k' - S(u_{k-1}'))$ *coincide for all* $\omega \in \Omega_N$.
- **(c)** *There is a number* $C_N > 0$ *such that for any integer* $r \ge 1$, *we have*[5]

  $$\mathbb{P}_N\big\{\mathsf{P}_N u_k = \mathsf{P}_N u_k' \text{ for } 1 \le k \le r-1, \mathsf{P}_N u_r \ne \mathsf{P}_N u_r'\big\} \le C_N \gamma_N^{r-1}\|v - v'\|, \tag{4.2}$$

  where γ_N *is the number in Condition* (C), P_N *is the orthogonal projection onto* $\operatorname{span}\{e_1, \dots, e_N\}$ *in* H *and* $\mathsf{Q}_N = 1 - \mathsf{P}_N$.

Proof of Proposition 4.1. Without loss of generality, we can assume that f and V are non-negative functions on $\mathcal{A}(B)$.

Let us fix an initial point $(v, v') \in \mathcal{A}(B)\times\mathcal{A}(B)$ such that $\varkappa := \|v - v'\| \le 1$, a sufficiently large integer $N \ge 1$, and apply Proposition 4.2. Let (u_k, u_k') be the corresponding sequence. We denote by $A(r)$ the event on the left-hand side of (4.2), and

$$\tilde{A}(r) = \big\{\mathsf{P}_N u_k = \mathsf{P}_N u_k' \text{ for } 1 \le k \le r\big\}.$$

Then we have

$$\mathfrak{P}_k^V f(v) - \mathfrak{P}_k^V f(v') = \sum_{r=1}^{k} I_k^r + \tilde{I}_k, \tag{4.3}$$

[5] The relation $\mathsf{P}_N u_k = \mathsf{P}_N u_k'$ in (4.2) should be omitted for $r = 1$.

where

$$I_k^r = \mathbb{E}_N\big\{\mathbb{1}_{A(r)}\big(\Xi_k^V f(u_k) - \Xi_k^V f(u_k')\big)\big\},$$
$$\tilde I_k = \mathbb{E}_N\big\{\mathbb{1}_{\tilde A(k)}\big(\Xi_k^V f(u_k) - \Xi_k^V f(u_k')\big)\big\},$$

and $\mathbb{E}_N$ is the expectation corresponding to $\mathbb{P}_N$.

Step 1. Estimate for I_k^r. Let us show that

$$|I_k^r| \le C_N \|f\|_\infty \gamma_N^{r-1} e^{r\|V\|_\infty} \|\mathfrak{P}_k^V \mathbf{1}\|_{\mathcal{A}(B)}\, \varkappa, \qquad 1 \le r \le k. \tag{4.4}$$

Indeed, let $\mathcal{F}_k^N$ be the filtration generated by (u_k, u_k'). Taking the conditional expectation given $\mathcal{F}_r^N$, using the fact that f and V are non-negative, and carrying out some simple estimates, we derive

$$\begin{aligned} I_k^r \le \mathbb{E}_N\big\{\mathbb{1}_{A(r)} \Xi_k^V f(u_k)\big\} &\le \|f\|_\infty e^{r\|V\|_\infty} \mathbb{E}_N\big(\mathbb{1}_{A(r)} \mathfrak{P}_{k-r}^V \mathbf{1}(u_r)\big) \\ &\le \|f\|_\infty e^{r\|V\|_\infty} \|\mathfrak{P}_k^V \mathbf{1}\|_{\mathcal{A}(B)} \mathbb{P}_N\big\{A(r)\big\}. \end{aligned}$$

Using (4.2), we obtain (4.4).

Step 2. Squeezing. Before estimating $\tilde I_k$, let us show the following squeezing property on the event $\tilde A(k)$:

$$\|u_r - u_r'\| \le \gamma_N^r\, \varkappa, \quad 1 \le r \le k. \tag{4.5}$$

Indeed, using property (b) in Proposition 4.2, we get

$$\|u_r - u_r'\| = \|\mathsf{Q}_N(u_r - u_r')\| = \|\mathsf{Q}_N(S(u_{r-1}) - S(u_{r-1}'))\| \le \gamma_N\, \|u_{r-1} - u_{r-1}'\|.$$

Iterating this, we arrive at the required result.

Step 3. Estimate for $\tilde I_k$. Let us show that

$$|\tilde I_k| \le C_1\, (\gamma_N\, \|f\|_\infty + \gamma_N^k\, \|f\|_L)\, \|\mathfrak{P}_k^V \mathbf{1}\|_{\mathcal{A}(B)}\, \varkappa \tag{4.6}$$

for some number $C_1 = C_1(\|V\|_L) > 0$ not depending on N. Indeed,

$$\begin{aligned} \tilde I_k &= \mathbb{E}\big\{\mathbb{1}_{\tilde A(k)} \Xi_k^V \mathbf{1}(u_k)[f(u_k) - f(u_k')]\big\} \\ &\quad + \mathbb{E}\big\{\mathbb{1}_{\tilde A(k)} [\Xi_k^V \mathbf{1}(u_k) - \Xi_k^V \mathbf{1}(u_k')] f(u_k')\big\} =: J_{1,k} + J_{2,k}. \end{aligned} \tag{4.7}$$

We derive from (4.5),

$$|J_{1,k}| \le \mathbb{E}\big\{\mathbb{1}_{\tilde A(k)} \Xi_k^V \mathbf{1}(u_k) |f(u_k) - f(u_k')|\big\} \le \gamma_N^k\, \|f\|_L \|\mathfrak{P}_k^V 1\|_{\mathcal{A}(B)}\, \varkappa.$$

Similarly, as $V \in L_b(H)$,

$$\begin{aligned} |J_{2,k}| &\le \|f\|_\infty \mathbb{E}\big\{\mathbb{1}_{\tilde A(k)} |\Xi_k^V \mathbf{1}(u_k) - \Xi_k^V \mathbf{1}(u_k')|\big\} \\ &\le \|f\|_\infty \mathbb{E}\left\{\mathbb{1}_{\tilde A(k)} \Xi_k^V \mathbf{1}(u_k) \left[\exp\left(\sum_{n=1}^k |V(u_n) - V(u_n')|\right) - 1\right]\right\} \\ &\le \|f\|_\infty \left[\exp\big(\varkappa \gamma_N\, (1 - \gamma_N)^{-1} \|V\|_L\big) - 1\right] \|\mathfrak{P}_k^V \mathbf{1}\|_{\mathcal{A}(B)}. \end{aligned}$$

Combining the estimates for $J_{1,k}$ and $J_{2,k}$ with (4.7), we obtain (4.6).

Step 4. Substituting (4.4) and (4.6) into (4.3), we derive

$$\begin{aligned}
&|\mathfrak{P}_k^V f(v) - \mathfrak{P}_k^V f(v')| \\
&\qquad \le \left(\tilde{C}_N \|f\|_\infty \sum_{r=1}^{k} \gamma_N^{r-1} e^{r\|V\|_\infty} + C_1 \gamma_N^k \|f\|_L \right) \|\mathfrak{P}_k^V \mathbf{1}\|_{\mathcal{A}(B)} \varkappa \\
&\qquad \le \left(C \|f\|_\infty + c^k \|f\|_L \right) \|\mathfrak{P}_k^V \mathbf{1}\|_{\mathcal{A}(B)} \varkappa
\end{aligned}$$

for sufficiently large N. □

5 Applications

In this section, we present various corollaries of Theorems 1.1-1.3.

5.1 Existence and analyticity of the pressure function

We start with the existence of the pressure function.

Proposition 5.1. *Assume that Conditions* (A)-(D) *are fulfilled. Then the following limit (called pressure function) exists*

$$Q(V,u) = \lim_{k\to\infty} \frac{1}{k} \log \mathfrak{P}_k^V \mathbf{1}(u) \tag{5.1}$$

for any $V \in C(H)$ and $u \in H$. Moreover, this limit does not depend on u if we have one of the following properties:

(1) *The initial condition u belongs to $\mathcal{A}$.*

(2) $\mathrm{Osc}(V) \le \delta$*, where δ is the number in Theorem 1.2.*

(3) *Condition* (E) *is satisfied.*

The limit in (5.1) *is denoted by $Q(V)$ if one of the properties* (1)-(3) *is satisfied.*

Proof. First assume that $V \in L_b(H)$. If we have one of (1)-(3), then by (1.5),

$$\lambda_V^{-k} \mathfrak{P}_k^V \mathbf{1}(u) \to h_V(u) \quad \text{as } k \to \infty.$$

Taking the logarithm, we infer that $Q(V,u) = \log \lambda_V$. In the general case, we cannot use the multiplicative ergodicity, so we proceed differently. We use the following lemma which is established below.

Lemma 5.2. *Under Conditions* (A)-(D)*, for any $V \in L_b(H)$ and bounded set $B \subset H$, there is a number $C > 0$ such that*

$$C^{-1} \|\mathfrak{P}_k^V \mathbf{1}\|_B \le \|\mathfrak{P}_k^V \mathbf{1}\|_{\mathcal{A}(B)} \le C \|\mathfrak{P}_k^V \mathbf{1}\|_B, \qquad k \ge 1. \tag{5.2}$$

We take any $u \in H$ and apply (5.2) for $B = \{u\}$,

$$C^{-1}\,\mathfrak{P}_k^V \mathbf{1}(u) \le \|\mathfrak{P}_k^V \mathbf{1}\|_{\mathcal{A}(\{u\})} \le C\,\mathfrak{P}_k^V \mathbf{1}(u), \qquad k \ge 1. \tag{5.3}$$

Since the set $\mathcal{A}(\{u\})$ is invariant for (0.1), we have

$$\|\mathfrak{P}_{n+m}^V \mathbf{1}\|_{\mathcal{A}(\{u\})} \le \|\mathfrak{P}_n^V \mathbf{1}\|_{\mathcal{A}(\{u\})} \|\mathfrak{P}_m^V \mathbf{1}\|_{\mathcal{A}(\{u\})}, \quad m, n \ge 1.$$

This implies that the function $f : \mathbb{Z}_+ \to \mathbb{R}$, $f(k) = \log \|\mathfrak{P}_k^V \mathbf{1}\|_{\mathcal{A}(\{u\})}$ is subadditive, hence, by the Fekete lemma, the following limit exists

$$\lim_{k\to\infty} \frac{1}{k} \log \|\mathfrak{P}_k^V \mathbf{1}\|_{\mathcal{A}(\{u\})}.$$

Applying (5.3), we get the existence of limit (5.1).

Now let us assume that $V \in C(H)$. As $\mathcal{A}(\{u\})$ is compact in H, we can find a sequence $V_n \in L_b(H)$ such that $\|V - V_n\|_{\mathcal{A}(\{u\})} \to 0$ as $n \to \infty$. Then using the inequality

$$\left| \frac{1}{k} \log \mathfrak{P}_k^V \mathbf{1}(u) - \frac{1}{k} \log \mathfrak{P}_k^{V_n} \mathbf{1}(u) \right| \le \|V - V_n\|_{\mathcal{A}(\{u\})}, \quad k, n \ge 1,$$

we get the existence of limit (5.1) for any $V \in C(H)$ and $u \in H$. $\square$

Proof of Lemma 5.2. Using the Markov property and the fact that V is bounded on $\mathcal{A}(B)$, we get

$$\mathfrak{P}_k^V \mathbf{1}(u) \le C_1\, \mathbb{E}_u \left(\mathfrak{P}_{k-1}^V \mathbf{1}(u_1) \right) \le C_1\, \|\mathfrak{P}_{k-1}^V \mathbf{1}\|_{\mathcal{A}(1,B)} \le C_2\, \|\mathfrak{P}_k^V \mathbf{1}\|_{\mathcal{A}(B)}$$

for any $k \ge 1$ and $u \in B$. This proves the first inequality in (5.2). The proof of the second inequality relies on the uniform Feller property. We argue by contradiction. If this inequality is not true, then there is a sequence $k_n \to \infty$ such that

$$\frac{\|\mathfrak{P}_{k_n}^V \mathbf{1}\|_B}{\|\mathfrak{P}_{k_n}^V \mathbf{1}\|_{\mathcal{A}(B)}} \to 0 \quad \text{as } n \to \infty. \tag{5.4}$$

By Proposition 4.1, the sequence $\{\|\mathfrak{P}_k^V \mathbf{1}\|_{\mathcal{A}(B)}^{-1} \mathfrak{P}_k^V \mathbf{1}, k \ge 0\}$ is uniformly equicontinuous on $\mathcal{A}(B)$. The Arzelà–Ascoli theorem implies the existence of a subsequence of k_n, which is again denoted by k_n, and a non-negative function $g \in C(\mathcal{A}(B))$ such that

$$\frac{\mathfrak{P}_{k_n}^V \mathbf{1}}{\|\mathfrak{P}_{k_n}^V \mathbf{1}\|_{\mathcal{A}(B)}} \to g \quad \text{in } C(\mathcal{A}(B)) \text{ as } n \to \infty. \tag{5.5}$$

Clearly, $\|g\|_{\mathcal{A}(B)} = 1$. Hence there is an integer $m \ge 1$ and a point $v_* \in \mathcal{A}(m, B)$ such that $g(v_*) > 0$. From (5.5) and the Lebesgue theorem on dominated convergence it follows that

$$\frac{\mathfrak{P}_{k_n+m}^V \mathbf{1}}{\|\mathfrak{P}_{k_n}^V \mathbf{1}\|_{\mathcal{A}(B)}} \to \mathfrak{P}_m^V g \quad \text{in } C(\mathcal{A}(B)) \text{ as } n \to \infty,$$

where $\mathfrak{P}_m^V g(u_*) > 0$ for some $u_* \in B$. Therefore, for any sufficiently large $n \geq 1$,

$$\frac{\mathfrak{P}_{k_n}^V \mathbf{1}(u_*)}{\|\mathfrak{P}_{k_n}^V \mathbf{1}\|_{\mathcal{A}(B)}} \geq e^{-m\|V\|_{\mathcal{A}(B)}} \frac{\mathfrak{P}_{k_n+m}^V \mathbf{1}(u_*)}{\|\mathfrak{P}_{k_n}^V \mathbf{1}\|_{\mathcal{A}(B)}} \geq \frac{1}{2} e^{-m\|V\|_{\mathcal{A}(B)}} \mathfrak{P}_m^V g(u_*),$$

which contradicts (5.4) and proves (5.2). □

Combining convergence (1.5) with a well-known perturbation argument [29, 19], we prove the analyticity of the pressure function.

Proposition 5.3. *Assume that Conditions* (A)-(D) *are fulfilled and* $V \in L_b(H)$. *Then there is a number* $p > 0$ *such that the following limit exists*

$$Q(zV) = \lim_{k\to\infty} \frac{1}{k} \log \mathfrak{P}_k^{zV} \mathbf{1}(u) \tag{5.6}$$

for any $u \in H$ *and* $z \in D_p = \{x \in \mathbb{C} : |x| \leq p\}$. *Moreover, the map* $z \mapsto Q(zV)$ *is real-analytic in some neighborhood of the origin. If one of the properties* (1) *and* (3) *in Proposition 5.1 is satisfied, then this map exists and is real-analytic on* $\mathbb{R}$.

Proof. Let us denote by $L_{b,\mathbb{C}}(H)$ the complexification of the space $L_b(H)$ and by $\mathcal{L}$ the space of bounded linear operators from $L_{b,\mathbb{C}}(H)$ to $L_{b,\mathbb{C}}(H)$ endowed with the natural norm $\|\cdot\|_{\mathcal{L}}$. We consider the family $\{\mathfrak{P}_1^{zV} : z \in D_p\}$ in $\mathcal{L}$. It is straightforward to check that this is a *holomorphic family* in the sense of Section VII.1.1 in [17], p. 365. By exponential mixing (0.2), the operator $\mathfrak{P}_1 = \mathfrak{P}_1^{zV}$ with $z = 0$ has a simple isolated eigenvalue $\lambda_0 = 1$ corresponding to eigenvectors $h_0 = \mathbf{1}$ and $\mu_0 = \mu$. Let $\mathsf{P}_0 f = \langle f, \mu\rangle$ be the spectral projection associated with this eigenvalue. Clearly, the spectral radius of the operator $\mathfrak{P}_1(1 - \mathsf{P}_0)$ is less than $e^{-\alpha} < 1$.

By Kato's holomorphic perturbation theorem (see Theorems 1.7 and 1.8 in Section VII.1.3 in [17], p. 368–370), there is a number $p > 0$ such that the following property holds:

- The operator $\mathfrak{P}_1^{zV}$ has a simple eigenvalue λ_{zV} for any $z \in D_p$. Moreover, the maping $z \to (\lambda_{zV}, \mathsf{P}_{zV})$ is analytic on D_p, where P_{zV} is the spectral projection associated with λ_{zV}.

In particular, for sufficiently small $p > 0$, we have

$$\inf_{z\in D_p} |\lambda_{zV}| > \gamma := (e^{-\alpha} + 1)/2, \tag{5.7}$$

$$\sup_{z\in D_p} \|\mathfrak{P}_1^{zV}(1 - \mathsf{P}_{zV})\|_{\mathcal{L}} \leq \gamma, \tag{5.8}$$

$$M := \sup_{x\in S_\gamma,\, z\in D_p} \|(I - x\,\mathfrak{P}_1^{zV}(I - \mathsf{P}_{zV}))^{-1}\|_{\mathcal{L}} < \infty, \tag{5.9}$$

where $S_\gamma = \{x \in \mathbb{C} : |x| = \gamma^{-1}\}$. By the Cauchy integral formula, we have

$$(\mathfrak{P}_1^{zV}(I - \mathsf{P}_{zV}))^k = \frac{1}{k!}\frac{\partial^k}{\partial x^k}(I - x\,\mathfrak{P}_1^{zV}(I - \mathsf{P}_{zV}))^{-1}|_{x=0}$$
$$= \frac{1}{2\pi i}\int_{S_\gamma} x^{-k-1}(I - x\,\mathfrak{P}_1^{zV}(I - \mathsf{P}_{zV}))^{-1}\,\mathrm{d}x, \quad k \geq 1.$$

Combining this with (5.8) and (5.9), we see that

$$\|\mathfrak{P}_k^{zV} - \lambda_{zV}^k \mathsf{P}_{zV}\|_{\mathcal{L}} = \|(\mathfrak{P}_1^{zV}(I - \mathsf{P}_{zV}))^k\|_{\mathcal{L}} \leq M\gamma^k.$$

This and (5.7) readily imply limit (5.6).

The remaining assertions are proved using limit (1.5) for the potential z_0V and applying a similar perturbation argument for the family $\{\mathfrak{P}_1^{zV} : z \in D_p(z_0)\}$, where $z_0 \in \mathbb{R}$ is arbitrary and $D_p(z_0) = \{x \in \mathbb{C} : |x - z_0| \leq p\}$. □

In view of limit (5.6), we can apply Bryc's criterion (see Proposition 1 in [2]). We obtain immediately that for any $V \in L_b(H)$ with $\langle V, \mu\rangle = 0$ and any $u \in H$, the following central limit theorem holds

$$\mathcal{D}_u\left(\frac{1}{\sqrt{k}}\sum_{n=1}^{k} V(u_n)\right) \to N(0, \sigma_V), \quad k \to \infty,$$

where μ is the stationary measure of $(u_k, \mathbb{P}_u)$, $\mathcal{D}_u$ is the distribution of a random variable under the law $\mathbb{P}_u$, and $\sigma_V = \frac{\partial^2}{\partial\alpha^2}Q(\alpha V)|_{\alpha=0}$. See Section 4.1.3 in [21] for another proof of this result and [28] for an estimate for the rate of convergence.

5.2 Large deviations

In this section, we give some applications to large deviations principle (LDP). We use some standard terminology from the LDP theory (e.g., see [4, 5]). Recall that the *occupation measures* for the trajectories of (0.1) are defined by

$$\zeta_k = \frac{1}{k}\sum_{n=0}^{k-1}\delta_{u_n}.$$

In Theorem 1.3 in [14], a level-2 LDP is obtained for the family $\{\zeta_k\}$ in the case when the initial condition belongs to $\mathcal{A}$. In this section, we complete that theorem, by stating two results that establish LDP in the case of an arbitrary initial condition in H. The following theorem gives, in particular, a level-1 LDP of local type under the same conditions as in [14].

Theorem 5.4. *Let Conditions* (A)-(D) *be fulfilled. Then for any non-constant function $f \in L_b(H)$, there is $\varepsilon = \varepsilon(f) > 0$ and a convex function $I^f : \mathbb{R} \to \mathbb{R}_+$ such that, for any open subset O of the interval $(\langle f, \mu\rangle - \varepsilon, \langle f, \mu\rangle + \varepsilon)$ and $u \in H$, we have*

$$\lim_{k\to\infty}\frac{1}{k}\log \mathbb{P}_u\{\langle f, \zeta_k\rangle \in O\} = -\inf_{x\in O} I^f(x),$$

where μ is the stationary measure. This limit is uniform with respect to u in a bounded set of H. Moreover, if Condition (E) is also fulfilled, then $\varepsilon = +\infty$.

This theorem follows immediately from the analyticity of the pressure function established in Proposition 5.3 and a local version of the Gärtner–Ellis theorem (e.g., see Theorem A.5 in [16]).

A level-2 LDP holds in the whole space H, provided that Conditions (A)-(E) are fulfilled. Namely, we have the following result.

Theorem 5.5. *Let the assumptions* (A)-(E) *be fulfilled. Then, there is a convex function $I : \mathcal{P}(H) \to [0, +\infty]$ with compact level sets $\{I \leq M\}$ in H for any $M > 0$ and that is infinite outside $\mathcal{P}(\mathcal{A})$ such that for any random initial point u_0 whose law $\lambda = \mathcal{D}u_0$ has a bounded support in H, we have*

$$-\inf_{\sigma\in\dot{\Gamma}} I(\sigma) \leq \liminf_{k\to\infty} \frac{1}{k} \log \mathbb{P}_\lambda\{\zeta_k \in \Gamma\} \leq \limsup_{k\to\infty} \frac{1}{k} \log \mathbb{P}_\lambda\{\zeta_k \in \Gamma\} \leq -\inf_{\sigma\in\bar{\Gamma}} I(\sigma)$$

for any subset $\Gamma \subset \mathcal{P}(H)$, where $\dot{\Gamma}$ and $\bar{\Gamma}$ stand for its interior and closure, respectively. Moreover, the function I can be written as

$$I(\sigma) = \sup_{V\in C(\mathcal{A})} \left(\langle V, \sigma\rangle - Q(V)\right), \qquad \sigma \in \mathcal{P}(\mathcal{A}), \tag{5.10}$$

where $Q(V)$ is the pressure function defined in Proposition 5.1.

This result can be proved using Theorem 1.3 and literally repeating the arguments of the proof of Theorem 1.3 in [14] based on the application of the Kifer's criterion obtained in [18].

5.3 The SLLN time

In paper [28], a strong law of large numbers is obtained for system (0.1). More precisely, it is proved that for any $f \in L_b(H)$, $u \in H$, and $\varepsilon > 0$, the following inequality holds

$$\left|\frac{1}{k}\sum_{n=1}^{k} f(u_n) - \langle f, \mu\rangle\right| \leq C\, k^{-1/2+\varepsilon} \qquad \text{for } k \geq T,$$

where μ is the stationary measure of $(u_k, \mathbb{P}_u)$ and $T \geq 1$ is a random integer whose any moment is finite, i.e., $\mathbb{E}_u T^m < \infty$ for any $m \geq 1$. Here we show that this polynomial bound on T is optimal.

Proposition 5.6. *Under Conditions* (A)-(D)*, assume that for some non-constant function $f \in L_b(H)$ with $f(0) \neq 0$ and initial condition $u \in H$, there is a sequence r_k going to zero as $k \to \infty$ and a random integer $T \geq 1$ such that*

$$\left|\frac{1}{k}\sum_{n=1}^{k} f(u_n) - \langle f, \mu\rangle\right| \leq r_k \quad \text{for } k \geq T. \tag{5.11}$$

Then T has an infinite exponential moment, i.e.,

$$\mathbb{E}_u e^{\alpha T} = +\infty \qquad \text{for any } \alpha > 0.$$

Proof. Step 1: Contradiction argument. Suppose that for some $\alpha>0$ and $M>0$,

$$\mathbb{E}_u e^{\alpha T} \le M. \tag{5.12}$$

Let $\delta > 0$ be the number in Theorem 1.2. Up to multiplying f by a small positive constant and deducing another one, we may assume that $\mathrm{Osc}(f) \le \delta$, $\|f\|_\infty \le \alpha$, and $\langle f, \mu\rangle = 0$. Then, by (1.5),

$$e^{-Q(f)k}\mathbb{E}_u \exp\left(\sum_{n=1}^{k} f(u_n)\right) \to h_f(u) \qquad \text{as } k\to\infty, \tag{5.13}$$

where $Q(f) = \log \lambda_f$. In Step 2, we will show that, up to replacing f by $-f$, we have

$$Q(f) > 0. \tag{5.14}$$

We infer from (5.11) that

$$\sum_{n=1}^{k} f(u_n) \le \alpha T + \mathbb{1}_{\{k\ge T\}} \sum_{n=1}^{k} f(u_n) \le \alpha T + k\, r_k,$$

which, together with (5.12), implies

$$\mathbb{E}_u \exp\left(\sum_{n=1}^{k} f(u_n)\right) \le e^{k r_k}\mathbb{E}_u e^{\alpha T} \le M e^{k r_k}.$$

Combining this with (5.13), (5.14), and convergence $r_k \to 0$, we get $h_f(u) = 0$, which is a contradiction.

Step 2: Proof of (5.14). As $Q : L_b(H) \to \mathbb{R}$ is convex and $Q(\mathbf{0}) = 0$, up to replacing f by $-f$, we can assume[6] that $Q(f) \ge 0$. Let us suppose that $Q(f) = 0$. Then from (1.5) we conclude that

$$\sup_{k\ge1} \mathbb{E}_\mu \exp\left(\sum_{n=1}^{k} f(u_n)\right) < \infty,$$

where $\mathbb{E}_\mu$ is the expectation corresponding to the stationary measure. This implies that

$$k^{-1}\,\mathbb{E}_\mu \left(\sum_{n=0}^{k} f(u_n)\right)^2 \to 0 \quad \text{ as } k\to\infty.$$

Combining this with Proposition 5.7, we get $f \equiv 0$. This contradicts the assumption that f is non-constant and completes the proof of the proposition. □

[6]In fact $Q(f) \ge \langle f, \mu\rangle = 0$ for any $f \in L_b(H)$. This follows from (5.10) and $I(\mu) = 0$; see (4.6) of [14].

Proposition 5.7. *Under the conditions of Proposition 5.6, the following limit exists for any $f \in L_b(H)$ with $\langle f, \mu \rangle = 0$:*

$$k^{-1}\,\mathbb{E}_\mu \left(\sum_{n=0}^{k} f(u_n) \right)^2 \to \sigma_f^2 \qquad \textit{as } k \to \infty. \tag{5.15}$$

Moreover, if $f(0) \neq 0$, then $\sigma_f \neq 0$.

Proof. This result is a discrete-time version of Proposition 4.1.4 in [21], and the proof is essentially the same except that here we do not have irreducibility of the process. By the Markov property and the stationarity of μ, we have

$$\begin{aligned}
\mathbb{E}_\mu \left(\sum_{n=0}^{k} f(u_n) \right)^2 &= \mathbb{E}_\mu \sum_{n=0}^{k} \sum_{r=0}^{k} f(u_r) f(u_n) \\
&= 2 \sum_{r=0}^{k} \sum_{n=r}^{k} \mathbb{E}_\mu \left(f(u_r) \mathbb{E}_\mu (f(u_n) | \mathcal{F}_r) \right) - \mathbb{E}_\mu \left(\sum_{n=0}^{k} f^2(u_n) \right) \\
&= 2 \sum_{r=0}^{k} \sum_{n=r}^{k} \mathbb{E}_\mu \left(f(u_r) (\mathfrak{P}_{n-r} f)(u_r) \right) - (k+1) \langle f^2, \mu \rangle \\
&= 2 \sum_{r=0}^{k} \sum_{n=r}^{k} \langle f \mathfrak{P}_{n-r} f, \mu \rangle - (k+1) \langle f^2, \mu \rangle \\
&= 2 \sum_{n=0}^{k} (k+1-n) \langle f \mathfrak{P}_n f, \mu \rangle - (k+1) \langle f^2, \mu \rangle.
\end{aligned}$$

Dividing this relation by k and passing to the limit as $k \to \infty$, we get (5.15) with $\sigma_f^2 = 2\langle g f, \mu \rangle - \langle f^2, \mu \rangle$ and $g(u) = \sum_{n=0}^{\infty} \mathfrak{P}_n f(u)$ for $u \in H$. Note that, by exponential mixing (0.2), the function $g : H \to \mathbb{R}$ is well defined and bounded on bounded sets of H.

To prove the second part of the proposition, let us assume that $\sigma_f = 0$ and consider Gordin's martingale approximation

$$M_k := \sum_{n=0}^{\infty} \left(\mathbb{E}_u (f(u_n) | \mathcal{F}_k) - \mathbb{E}_u (f(u_n) | \mathcal{F}_0) \right), \quad u \in H, \quad k \geq 0,$$

where $\mathcal{F}_k$ is the filtration corresponding to the Markov process $(u_k, \mathbb{P}_u)$. We shall use the following equality for these approximations [7]

$$\sum_{n=0}^{k-1} f(u_n) = M_k - g(u_k) + g(u). \tag{5.16}$$

Repeating the arguments of (4.19) in [21], we see that

$$\mathbb{P}_\mu\{M_k = 0 \quad \text{for all } k \geq 0\} = 1. \tag{5.17}$$

Let us show that this equality implies that $f \equiv 0$. Indeed, assume that $f(0) > 0$ (the case $f(0) < 0$ is similar), and let B be a ball in H centred at zero such that $f(u) > \varepsilon$ for any $u \in B$. Using the facts that $S(0) = 0$ (which follows from Condition (A)), $0 \in \mathcal{K}$ (see Condition (D)), and $0 \in \operatorname{supp} \mu$ (which follows from (1.1) and Condition (D)), we see that

$$\mathbb{P}_\mu\{u_n \in B : n = 0, \dots, k\} > 0 \quad \text{for any } k \geq 0.$$

This implies that

$$\mathbb{P}_\mu\left\{\sum_{n=0}^{k-1} f(u_n) > k\varepsilon\right\} > 0 \quad \text{for any } k \geq 1. \tag{5.18}$$

Let $C > 0$ be such that $|g(u)| \leq C$ for $u \in B$. Then $|g(u_k) - g(u)| \leq 2C$ if $u_k, u \in B$. If $k \geq 1$ is so large that $k\varepsilon > 2C$, combining (5.16) and (5.18), we see that (5.17) cannot hold. This contradiction shows that $\sigma_f \neq 0$. ☐

5.4 The speed of attraction

For any $\varepsilon > 0$, let us introduce the random variable

$$\mathcal{N}^\varepsilon(\omega) = \#\{m \geq 1 : u_m(\omega) \notin \mathcal{A}_\varepsilon\},$$

where, as before, $\mathcal{A}_\varepsilon$ is the ε-neighborhood of $\mathcal{A}$ in H.

Proposition 5.8. *Let Conditions* (A)-(D) *be fulfilled. Then, for any $\varepsilon > 0$ and $u \in H$, the random variable $\mathcal{N}^\varepsilon(\omega)$ is $\mathbb{P}_u$-almost surely finite. Moreover, there is a positive constant α not depending on ε such that*

$$\sup_{u \in B_R} \mathbb{E}_u e^{\alpha \mathcal{N}^\varepsilon} < \infty \qquad \textit{for any } R > 0. \tag{5.19}$$

If in addition Condition (E) *is satisfied, then* (5.19) *holds with any $\alpha > 0$.*

[7] Equality (5.16) follows immediately from the Markov property:

$$\begin{aligned} M_k &= \sum_{n=0}^{k-1} f(u_n) + \sum_{n=k}^{\infty} \mathbb{E}_u(f(u_n)|\mathcal{F}_k) - \sum_{n=0}^{\infty} \mathbb{E}_u(f(u_n)|\mathcal{F}_0) \\ &= \sum_{n=0}^{k-1} f(u_n) + \sum_{n=k}^{\infty} (\mathfrak{P}_{n-k} f)(u_k) - \sum_{n=0}^{\infty} (\mathfrak{P}_n f)(u) \\ &= \sum_{n=0}^{k-1} f(u_n) + g(u_k) - g(u). \end{aligned}$$

Proof. Let $\delta > 0$ be the number entering Lemma 3.3. The proposition will be established if we show that there is a positive constant Λ depending on δ, ε, and R such that

$$\mathbb{P}_u\{\mathcal{N}^\varepsilon \geq m\} \leq \Lambda e^{-\delta m} \qquad \text{for any } u \in B_R \text{ and } m \geq 1. \tag{5.20}$$

Indeed, then inequality (5.19) will hold with $\alpha = \delta/2$. To this end, we consider a function $V \in L_b(H)$ that vanishes on $\mathcal{A}$, equals δ outside $\mathcal{A}_\varepsilon$, and satisfies $0 \leq V \leq \delta$ on H. It follows that $\mathrm{Osc}(V) \leq \delta$. Moreover, since V vanishes on $\mathcal{A}$, we have $\lambda_V = 1$ so that inequality (3.4) holds true for this V. Let us introduce the random variable

$$\mathcal{N}_k^\varepsilon(\omega) = k \wedge \mathcal{N}^\varepsilon(\omega)$$

and note that

$$\mathfrak{P}_k^V \mathbf{1}(u) \geq \mathbb{E}_u \left\{ \mathbb{1}_{\{\mathcal{N}_k^\varepsilon \geq m\}} \exp\left(\sum_{n=1}^k V(u_n) \right) \right\} \geq e^{\delta m} \mathbb{P}_u\{\mathcal{N}_k^\varepsilon \geq m\}.$$

Letting k go to infinity and using (3.4), we arrive at (5.20). Now if Condition (E) is also fulfilled, by Lemma 3.5, the above $\delta > 0$ can be chosen arbitrarily large and thus so can be α. $\square$

5.5 Kick-forced PDEs

Theorems 1.1-1.3 can be applied to a large class of dissipative PDEs perturbed by a *random kick force*. In this section, we discuss the validity of Conditions (A)-(E) for the Navier–Stokes, the complex Ginzburg–Landau, and the Burgers equations.

5.5.1 2D Navier–Stokes system

Let us consider the 2D Navier–Stokes (NS) system for incompressible fluids:

$$\partial_t u + \langle u, \nabla \rangle u - \nu \Delta u + \nabla p = \eta(t, x), \quad \mathrm{div}\, u = 0, \quad x \in D, \tag{5.21}$$

where $D \subset \mathbb{R}^2$ is a bounded domain with smooth boundary ∂D, $\nu > 0$ is the viscosity, $u = (u_1, u_2)$ and p are unknown velocity field and pressure, η is an external random force, and $\langle u, \nabla \rangle = u_1 \partial_1 + u_2 \partial_2$. We denote by H the L^2-space of divergence free vector fields

$$H = \left\{ u \in L^2(D, \mathbb{R}^2) : \mathrm{div}\, u = 0 \text{ in } D, \langle u, \boldsymbol{n} \rangle = 0 \text{ on } \partial D \right\}$$

endowed with the norm $\|\cdot\|$, where $\boldsymbol{n}$ is the outward unit normal to ∂D. By projecting (5.21) to H, we eliminate the pressure and obtain an evolution system for the velocity field (e.g., see Section 6 in Chapter 1 of [23]):

$$\partial_t u + B(u) + \nu L u = \Pi \eta(t, x), \tag{5.22}$$

where Π is the orthogonal projection onto H in L^2 (i.e., the Leray projection), $L = -\Pi\Delta$ is the Stokes operator, and $B(u) = \Pi(\langle u, \nabla \rangle u)$. We assume that η is a random kick force of the form

$$\eta(t, x) = \sum_{k=1}^{\infty} \delta(t - k)\eta_k(x), \tag{5.23}$$

where δ is the Dirac measure concentrated at zero and η_k are i.i.d. random variables in H satisfying Condition (D) with respect to an orthonormal basis $\{e_j\}$ formed by the eigenvectors of L. Under these assumptions, the trajectory u_t of (5.22) is normalised to be right-continuous and it is completely determined by its restriction u_k to integer times. If we denote by $S : H \to H$ the time-1 shift along the trajectories of (5.22) with $\eta = 0$, then the sequence $\{u_k\}$ satisfies (0.1). The validity of Conditions (A)-(C) for this system is checked in Section 3.2.4 in [21].

Proposition 5.9. *There is a number $\nu_* > 0$ such that for $\nu \geq \nu_*$, Condition* (E) *is satisfied for the NS system with the metric inherited from H.*

Remark 5.10. Let us note that in the case of large viscosity ν, the ergodicity of the Markov process $(u_k, \mathbb{P}_u)$ associated with (5.21) has a quite simple proof; see Exercice 2.5.9 in [21]. It seems however, that this assumption does not lead to an easy proof of the multiplicative ergodic theorem due to the presence of the potential V, which under condition (E) can have an arbitrarily large oscillation.

Proof of Proposition 5.9. We split the proof into two steps.

Step 1. Let us first show that the number ρ in Condition (B) can be chosen the same for any $\nu \geq 1$. Indeed, using the inequality

$$\|S(u_0)\| \leq e^{-\alpha_1 \nu}\|u_0\| \quad \text{for } u_0 \in H$$

and the fact that $\mathbb{P}\{\|\eta_1\| \leq C\} = 1$, we get

$$\|u_1\| \leq e^{-\alpha_1 \nu}\|u_0\| + C$$

with probability 1, where α_1 is the first eigenvalue of L and $C > 0$ does not depend on ν. This shows that the ball B_ρ of radius $\rho \geq C(1 - e^{-\alpha_1 \nu})^{-1}$ is invariant for (0.1). Moreover, if we choose $\rho \geq 2C(1 - e^{-\alpha_1 \nu})^{-1}$, then (1.2) is satisfied. We take $\rho = 2C(1 - e^{-\alpha_1})^{-1}$.

Step 2. Let us show that

$$\|S(u_0) - S(v_0)\| \leq \|u_0 - v_0\| \quad \text{for } u_0, v_0 \in \mathcal{A}(B_{\rho+1})$$

if ν is sufficiently large. We denote by u and v the solutions of (5.22) issued from u_0 and v_0, respectively. Then $w = u - v$ satisfies

$$\dot{w} + B(w, u) + B(v, w) + \nu L w = 0, \tag{5.24}$$

where $B(w,u) = \Pi(\langle w, \nabla \rangle u)$. Taking the scalar product of (5.24) with $2w$ in H, using the equality $\langle B(v,w), w\rangle = 0$ and the estimate

$$|\langle B(w,u), w\rangle| \le C_1 \|u\|_1 \|w\|_1^2,$$

where $\|\cdot\|_1$ is the norm in the Sobolev space $H^1(D,\mathbb{R}^2)$, we get

$$\partial_t \|w\|^2 + 2(\nu - C_1\|u\|_1)\|w\|_1^2 \le 0,$$

where C_1 does not depend on ν. By taking the scalar product of (5.22) for $\eta = 0$ with $2u$ in H, it is easy to see that

$$C_2 := \sup_{\nu \ge 1,\, u_0 \in \mathcal{A}(B_{\rho+1})} \int_0^1 \|u(s)\|_1 \, \mathrm{d}s < \infty.$$

Using the previous two inequalities together with the Poincaré inequality and the Gronwall lemma, we infer

$$\begin{aligned}\|w(1)\|^2 &\le \|w(0)\|^2 \exp\left(-2\alpha_1\nu + C_1\alpha_1 \int_0^1 \|u(s)\|_1 \, \mathrm{d}s\right) \\ &\le \|w(0)\|^2 \exp\left(-2\alpha_1\nu + C_1\alpha_1 C_2\right) \le \|w(0)\|^2\end{aligned}$$

for $\nu \ge \nu_* := C_1C_3/2$. Note that we even have a contraction for sufficiently large ν. □

5.5.2 Complex Ginzburg–Landau equation

The situation is similar for the complex Ginzburg–Landau (CGL) equation:

$$\partial_t u - (\nu + i)\Delta u + ia|u|^2 u = \eta(t,x), \quad x \in D, \quad u\big|_{\partial D} = 0, \tag{5.25}$$

where $\nu, a > 0$ are some numbers, $D \subset \mathbb{R}^3$ is a bounded domain with smooth boundary ∂D, $u = u(t,x)$ is a complex-valued function, and η is a kick force of the form (5.23). We consider this equation in the complex space $H = H_0^1(D)$ endowed with the norm $\|\cdot\|_1$. The random variables η_k are assumed to be i.i.d. in H and of the form

$$\eta_k(x) = \sum_{j=1}^{\infty} b_j(\xi_{jk}^1 + i\xi_{jk}^2)e_j(x),$$

where $\{e_j\}$ is an orthonormal basis in H formed by the eigenvectors of the Dirichlet Laplacian, $b_j > 0$ for all $j \ge 1$, and ξ_{jk}^i are independent real-valued random variables whose laws possess the properties stated in Condition (D). By Proposition 1.7 in [14], Conditions (A)-(C) hold for the CGL equation. The following result is an analogue of Propositions 5.9.

Proposition 5.11. *There is a number $\nu_* > 0$ such that for $\nu \ge \nu_*$, Condition* (E) *is satisfied for the CGL equation with the metric inherited from H.*

Sketch of the proof. Inequalities (1.36) and (1.37) in [14], combined with the arguments of Step 1 of the previous proof, show that the number ρ in Condition (B) can be chosen the same for any $\nu \geq 1$. We check that for sufficiently large ν,

$$\|S(u_0) - S(v_0)\|_1 \leq \|u_0 - v_0\|_1 \quad \text{for } u_0, v_0 \in \mathcal{A}(B_{\rho+1}),$$

where $S : H \to H$ is the time-1 shift for (5.25) with $\eta = 0$. Let u and v be the solutions of (5.25) issued from u_0 and v_0. Then $w = u - v$ satisfies

$$\partial_t w - (\nu + i)\Delta w + ia(|u|^2 u - |v|^2 v) = 0.$$

Multiplying this equation by $2\,\Delta w$ and integrating, we get

$$\begin{aligned} \partial_t \|w\|_1^2 &= 2\mathrm{Re} \int_D \nabla \dot{w} \cdot \nabla \bar{w} \,\mathrm{d}x = -2\mathrm{Re} \int_D \dot{w}\, \Delta \bar{w} \,\mathrm{d}x \\ &= -2\mathrm{Re} \int_D ((\nu + i)\Delta w - ia(|u|^2 u - |v|^2 v))\, \Delta \bar{w} \,\mathrm{d}x \\ &\leq -2\nu \,\|\Delta w\|^2 + 2a \,\big\||u|^2 u - |v|^2 v\big\|\, \|\Delta w\|, \end{aligned} \tag{5.26}$$

where $\|\cdot\|$ is the L^2-norm. The Hölder inequality and the embedding $H_0^1 \subset L^6$ imply that

$$\big\||u|^2 u - |v|^2 v\big\| \leq C_1 \big(\|u\|_1 + \|v\|_1\big)^2 \|w\|_1.$$

Substituting this into (5.26) and using the Poincaré inequality, we obtain

$$\partial_t \|w\|_1^2 \leq -\big(\nu\alpha_1 - C_1(\|u\|_1 + \|v\|_1)^4\big) \|w\|_1^2.$$

The Gronwall lemma and the standard inequality

$$C_2 := \sup_{\nu \geq 1,\, t \in [0,1],\, u_0 \in \mathcal{A}(B_{\rho+1})} \|u(t)\|_1 \,\mathrm{d}s < \infty$$

imply that

$$\begin{aligned} \|w(1)\|_1^2 &\leq \exp\left(-\nu\alpha_1 + C_1 \int_0^1 (\|u_1\|_1 + \|u_2\|_1)^4 \,\mathrm{d}s\right) \|w(0)\|_1^2 \\ &\leq \exp\big(-\nu\alpha_1 + 16\, C_1 C_2^4\big) \|w(0)\|_1^2 \leq \|w(0)\|_1^2 \end{aligned}$$

for $\nu \geq \nu_* := 16\, C_1 C_2^4 / \alpha_1$. □

5.5.3 Burgers equation

Let us consider the Burgers equation on the circle $\mathbb{S} = \mathbb{R}/2\pi\mathbb{Z}$:

$$\partial_t u - \nu \partial_x^2 u + u \partial_x u = \eta(t, x),$$

where $\nu > 0$ and η is of the form (5.23) with i.i.d. random variables $\{\eta_k\}$ in

$$H = \left\{ u \in L^2(\mathbb{S}, \mathbb{R}) : \int_{\mathbb{S}} u(x) \,\mathrm{d}x = 0 \right\}$$

satisfying Condition (D) with an orthonormal basis $\{e_j\}$ formed by the eigenvectors of the periodic Laplacian. The verification of Conditions (A)-(C) is similar to the case of the NS system.

Proposition 5.12. *Condition* (E) *is satisfied for the Burgers equation with any* $\nu > 0$ *with the metric inherited from* $L^1(\mathbb{S}, \mathbb{R})$.

This proposition follows immediately from inequality

$$\|S(u_0) - S(v_0)\|_{L^1(\mathbb{S},\mathbb{R})} \leq \|u_0 - v_0\|_{L^1(\mathbb{S},\mathbb{R})} \quad \text{for any } u_0, v_0 \in H \text{ and } \nu > 0$$

established in Section 3.3 of [12].

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