Approximate controllability of nonlinear parabolic PDEs in arbitrary space dimension

Vahagn Nersesyan[1]

June 12, 2020

Abstract

In this paper, we consider a parabolic PDE on a torus of arbitrary dimension. The nonlinear term is a smooth function of polynomial growth of any degree. In this general setting, the Cauchy problem is not necessarily well posed. We show that the equation in question is approximately controllable by only a finite number of Fourier modes. This result is proved by using some ideas from the geometric control theory introduced by Agrachev and Sarychev.

AMS subject classifications: 35K55, 93B05

Keywords: Parabolic PDEs, approximate controllability, geometric control theory

Contents

[1]Université Paris-Saclay, UVSQ, CNRS, Laboratoire de Mathématiques de Versailles, 78000, Versailles, France & Centre de Recherches Mathématiques, CNRS UMI 3457, Université de Montréal, Montréal, QC, H3C 3J7, Canada; e-mail: Vahagn.Nersesyan@math.uvsq.fr

0 Introduction

We consider the following parabolic PDE on the d-dimensional torus $\mathbb{T}^d$:

$$\partial_t u - \nu\Delta u + f(u) = h(t,x) + \eta(t,x), \quad (t,x) \in (0,T) \times \mathbb{T}^d,\ d \geq 1, \tag{0.1}$$

where ν is a positive number, $h : [0,T] \times \mathbb{T}^d \to \mathbb{R}$ is a given smooth function, and $f : \mathbb{R} \to \mathbb{R}$ is a nonlinear term. The latter is assumed to be of the form

$$f(y) = P_p(y) + g(y), \tag{0.2}$$

where P_p is a polynomial of degree $p \geq 2$ and $g : \mathbb{R} \to \mathbb{R}$ is a bounded smooth function with bounded derivatives. Let us emphasise that the parameters d, p, and T are arbitrary, so that Eq. (0.1) supplemented with the initial condition

$$u(0) = u_0 \tag{0.3}$$

is not necessarily well posed on the time interval $J_T := [0,T]$. For example, see Section 17 in the book [QS07] and the references therein for constructions of finite time blow-up solutions for problem (0.1), (0.3). We shall take initial condition in the Sobolev space $H^s(\mathbb{T}^d)$ with $s > d/2$, in order to have locally well-posed Cauchy problem, i.e., local existence, uniqueness, and continuous dependence on the initial condition and the source term (see Proposition 1.1).

The function η plays the role of the control and is assumed to be *degenerate* in the Fourier space. More precisely, η takes values in a finite-dimensional space defined by

$$\mathcal{H}(\mathcal{I}) = \operatorname{span}\{\sin\langle x,k\rangle, \cos\langle x,k\rangle : k \in \mathcal{I}\}, \tag{0.4}$$

where $\mathcal{I} \subset \mathbb{Z}^d$ is a finite symmetric set (i.e., $\mathcal{I} = -\mathcal{I}$) containing the origin and $\langle\cdot,\cdot\rangle$ is the Euclidean scalar product in $\mathbb{R}^d$. Recall that $\mathcal{I}$ is called a generator if any element of $\mathbb{Z}^d$ is a linear combination of elements of $\mathcal{I}$ with integer coefficients.

To formulate the main result of this paper, let us fix any $s > d/2$, $T > 0$, and $h \in L^2(J_T, H^{s-1}(\mathbb{T}^d))$. We shall say that Eq. (0.1) is *approximately controllable* by $\mathcal{H}(\mathcal{I})$-valued control if for any initial point $u_0 \in H^s(\mathbb{T}^d)$, any target $u_1 \in H^s(\mathbb{T}^d)$, and any number $\varepsilon > 0$, there is a control $\eta \in L^2(J_T, \mathcal{H}(\mathcal{I}))$ and a unique solution u of problem (0.1), (0.3) defined on the interval J_T such that

$$\|u(T) - u_1\|_{H^s} < \varepsilon.$$

Main Theorem. *Assume that one of the following two conditions hold: a) $g = 0$; or b) $p > s > d/2$ and g is arbitrary function as above. If $\mathcal{I}$ is a generator, then Eq.* (0.1) *is approximately controllable by $\mathcal{H}(\mathcal{I})$-valued control.*

The interpretation of the part b) is that a sufficiently strong polynomial component is needed in the nonlinearity in order to brake the influence of the perturbation g. See Section 2 for more general results. In particular, in the case when f is a polynomial (i.e., $g = 0$) and $h = 0$, the condition that the set $\mathcal{I}$ is a generator is also necessary for approximate controllability (see Theorem 2.6).

Note that the condition on $\mathcal{I}$ is completely independent of the choice of the functions f and h and the parameters ν, s, p, and T.

The proof of the Main Theorem uses some arguments from the works of Agrachev and Sarychev [AS05, AS06, AS08], who studied the approximate controllability of the 2D Navier–Stokes (NS) and Euler systems by finite-dimensional forces. Their approach has been extended to different equations by many authors. Shirikyan [Shi06, Shi07] established the approximate controllability of the 3D NS system on the torus. He also considered the Burgers equation on the real line in [Shi14] and on a bounded interval with Dirichlet boundary conditions in [Shi18]. Rodrigues [Rod06] proved approximate controllability of the 2D NS system on a rectangle with Lions boundary conditions, and with Phan [PR18] they generalised that result to the 3D case. In the papers [Ner10, Ner11], Nersisyan considered 3D Euler system for incompressible and compressible fluids, and Sarychev [Sar12] considered the 2D defocusing cubic Schrödinger equation. The controllability of the Lagrangian trajectories of the 3D NS system is considered in [Ner15] by the author.

We use a technique of applying large controls on small time intervals inspired by the works of Jurdjevic and Kupka (see the paper [JK85] and Chapter 5 in the book [Jur97]), who considered finite-dimensional control systems. Infinite-dimensional generalisations of this approach appear in the above-mentioned papers of Agrachev and Sarychev (e.g., see Section 6.2 in [AS06]) and in the paper [GHHM18] of Glatt-Holtz, Herzog, and Mattingly. In the latter, the authors prove, in particular, approximate controllability of a 1D parabolic PDE with polynomial nonlinearity of odd degree.

Without going into the technical details, let us describe some ideas of the proof of the Main Theorem. Together with Eq. (0.1), we consider an equation of the form

$$\partial_t u - \nu\Delta(u+\zeta) + f(u+\zeta) = h(t,x) + \eta(t,x) \tag{0.5}$$

with ζ and η taking values in $\mathcal{H}(\mathcal{I})$. It turns out that Eq. (0.1) is approximately controllable with control η if and only if so is Eq. (0.5) with two controls ζ and η. The solution u of problem (0.5), (0.3), whenever exists, is denoted by

$$\mathcal{R}_t(u_0, \zeta, h+\eta) := u(t). \tag{0.6}$$

The first step is the following asymptotic property that holds for any smooth functions ζ and η not depending on time:

$$\mathcal{R}_\delta(u_0, \delta^{-1/p}\zeta, h+\delta^{-1}\eta) \to u_0 + \eta - c\zeta^p \quad \text{in } H^s(\mathbb{T}^d) \text{ as } \delta \to 0^+, \tag{0.7}$$

where c is the leading coefficient of the polynomial P_p in (0.2). This allows to steer the trajectory of (0.5), (0.3) in small time close to any target u_1 belonging to the affine space $u_0 + \mathcal{H}_1(\mathcal{I})$, where $\mathcal{H}_1(\mathcal{I})$ is the largest vector space whose elements can be written in the form

$$\eta - \sum_{m=1}^{n} \zeta_m^p \tag{0.8}$$

for some integer $n \geq 1$ and vectors $\eta, \zeta_1, \ldots, \zeta_n \in \mathcal{H}(\mathcal{I})$ (see Section 2 for the precise definition of $\mathcal{H}_1(\mathcal{I})$). Then iterating this argument, we show that starting from u_0 we can also attain approximately any point in $u_0 + \mathcal{H}_2(\mathcal{I})$, where the space $\mathcal{H}_2(\mathcal{I})$ is defined by (0.8), but now with vectors $\eta, \zeta_1, \ldots, \zeta_n \in \mathcal{H}_1(\mathcal{I})$. In this way, we construct a non-decreasing sequence of subspaces $\{\mathcal{H}_j(\mathcal{I})\}$ such that the points in $u_0 + \mathcal{H}_j(\mathcal{I})$ are attainable from u_0. From the fact that $\mathcal{I}$ is a generator we deduce that the union $\cup_{j=1}\mathcal{H}_j(\mathcal{I})$ is dense in $H^s(\mathbb{T}^d)$ (i.e., $\mathcal{H}(\mathcal{I})$ is *saturating* in the language of the geometric control theory). This allows to control approximately Eq. (0.5) to any point in $H^s(\mathbb{T}^d)$ in small time. The controllability in any time T is derived by steering the system close to the target u_1 in small time, then by keeping the trajectory close to u_1 for a sufficiently long period of time, by applying an appropriate control.

The main contribution of this paper is the generality of the assumptions on the nonlinear term. We give a simple condition on the set of the Fourier modes $\mathcal{I}$ that ensures the controllability of the system. Surprisingly, the condition is independent of the nonlinear term and is also necessary in the polynomial case. The perturbation term g in (0.2) brings new difficulties that do not appear in the previously considered situations. Indeed, the nonlinear term f now may vanish on some ball, so it will not be able to couple there the Fourier modes in $\mathcal{I}$ to the others. This situation seems to be out of the reach with previous methods. The main ingredient in the above scheme of the proof is the limit (0.7). According to that limit, when applying large controls $\delta^{-1/p}\zeta$ on small time intervals $[0, \delta]$, we only see the highest order term cy^p of the nonlinearity $f(y)$ when we pass to the limit $\delta \to 0^+$. Thus the contribution of the perturbation g vanishes in the limit, provided that the condition b) is satisfied.

The proof of approximate controllability we give is short and conceptually simple; it is quite general and can be adapted to more degenerate problems. In [BGN20], this approach is further developed to consider the controllability of the system of 3D primitive equations of geophysical fluid dynamics with control acting directly only on the temperature equation.

Finally, let us mention that this paper is partially motivated by applications to the ergodicity of randomly forced PDEs. Indeed, the control theory is known to be a useful tool in the study of stochastic systems with highly degenerate noise. We refer the reader to the paper [KNS19] for more details and references on this subject and for a concrete application of our Main Theorem in the study of a stochastic version of Eq. (0.1), i.e., when the control η is replaced by a random process.

This paper is organised as follows. In Section 1, we establish a perturbative result on the existence and stability of solutions of problem (0.1), (0.3). The proof of the Main Theorem is given in Section 2. In Section 3, we prove limit (0.7), and in Section 4, we construct examples of saturating spaces.

Acknowledgement

The author thanks the referees for their valuable comments and suggestions. The author thanks Armen Shirikyan for helpful discussions. This research was supported by the *Agence Nationale de la Recherche* through the grant NONSTOPS ANR-17-CE40-0006-02 and by the CNRS through the PICS grant *Fluctuation theorems in stochastic systems*.

Notation

In this paper, we use the following notation.

$\mathbb{Z}^d$ is the integer lattice in $\mathbb{R}^d$.

$\mathbb{T}^d$ is the standard d-dimensional torus $\mathbb{R}^d/2\pi\mathbb{Z}^d$.

$H^s := H^s(\mathbb{T}^d)$ is the Sobolev space of order s endowed with the usual norm $\|\cdot\|_s$.

Let X be a Banach space endowed with the norm $\|\cdot\|_X$ and let $J_T := [0,T]$.

$B_X(a,r)$ is the closed ball of radius $r > 0$ centred at $a \in X$. We write $B_X(r)$, when $a = 0$.

$L^q(J_T, X)$, $1 \le q < \infty$ is the space of measurable functions $u : J_T \to X$ endowed with the norm

$$\|u\|_{L^q(J_T,X)} := \left(\int_0^T \|u(t)\|_X^q \,\mathrm{d}t\right)^{1/q} < \infty.$$

$L^q_{loc}(\mathbb{R}_+, X)$ is the space of measurable functions $u : \mathbb{R}_+ \to X$ whose restriction to the interval J_T belongs to $L^q(J_T, X)$ for any $T > 0$.

$C(J_T, X)$ is the space of continuous functions $u : J_T \to X$ with the norm

$$\|u\|_{C(J_T,X)} := \max_{t\in J_T} \|u(t)\|_X.$$

$x \wedge y$ denotes the minimum of real numbers x and y.

C, C_1, ... denote some unessential positive constants.

1 Local well-posedness and stability

In this section, we study the local existence and stability of solutions for the following generalisation of Eq. (0.1):

$$\partial_t u - \nu\Delta(u + \zeta) + f(u + \zeta) = \varphi, \tag{1.1}$$

where f is of the form (0.2) with any polynomial P_p of degree $p \ge 2$ and any bounded smooth function $g : \mathbb{R} \to \mathbb{R}$ with bounded derivatives. In this paper, by smooth we always mean C^∞-smooth. For any $T > 0$ and any integer $s > d/2$, we define the space

$$\mathcal{X}_{T,s} := C(J_T, H^s) \cap L^2(J_T, H^{s+1})$$

and endow it with the norm

$$\|u\|_{\mathcal{X}_{T,s}} := \|u\|_{C(J_T,H^s)} + \|u\|_{L^2(J_T,H^{s+1})}.$$

Proposition 1.1. *Let $\hat{u}_0 \in H^s$, $\hat{\zeta} \in C(\mathbb{R}_+, H^{s+1})$, and $\hat{\varphi} \in L^2_{loc}(\mathbb{R}_+, H^{s-1})$. There is a maximal time $T_* := T_*(\hat{u}_0, \hat{\zeta}, \hat{\varphi}) > 0$ and a unique solution $\hat{u}$ of problem (1.1), (0.3) with $(u_0, \zeta, \varphi) = (\hat{u}_0, \hat{\zeta}, \hat{\varphi})$ whose restriction to J_T belongs to $\mathcal{X}_{T,s}$ for any $T < T_*$. If $T_* < \infty$, then*

$$\|\hat{u}(t)\|_s \to +\infty \quad \text{as } t \to T_*^-. \tag{1.2}$$

Furthermore, for any $T < T_*$, there are positive constants $\delta = \delta(T, \Lambda)$ and $C = C(T, \Lambda)$, where

$$\Lambda := \|\hat{\zeta}\|_{C(J_T, H^{s+1})} + \|\hat{\varphi}\|_{L^2(J_T, H^{s-1})} + \|\hat{u}\|_{\mathcal{X}_{T,s}}, \tag{1.3}$$

such that the following properties hold.

(i) For any $u_0 \in H^s$, $\zeta \in C(J_T, H^{s+1})$, and $\varphi \in L^2(J_T, H^{s-1})$ satisfying

$$\|u_0 - \hat{u}_0\|_s + \|\zeta - \hat{\zeta}\|_{C(J_T, H^{s+1})} + \|\varphi - \hat{\varphi}\|_{L^2(J_T, H^{s-1})} < \delta, \tag{1.4}$$

problem (1.1), (0.3) has a unique solution $u \in \mathcal{X}_{T,s}$.

(ii) As in (0.6), let $\mathcal{R}$ be the resolving operator for (1.1), i.e., the mapping taking a triple (u_0, ζ, φ) satisfying (1.4) to the solution u. Then

$$\|\mathcal{R}(u_0, \zeta, \varphi) - \mathcal{R}(\hat{u}_0, \hat{\zeta}, \hat{\varphi})\|_{\mathcal{X}_{T,s}} \le C\big(\|u_0 - \hat{u}_0\|_s + \|\zeta - \hat{\zeta}\|_{C(J_T, H^{s+1})} + \|\varphi - \hat{\varphi}\|_{L^2(J_T, H^{s-1})}\big). \tag{1.5}$$

Proof. Local existence of a solution $\hat{u}$ and (1.2) are proved using a fixed point approach based on estimates that we will use in the proof of (i) and (ii) below. As the argument is quite standard, we skip the details of the proof of that part.

Step 1. Proof of (i). Let $T < T_*(\hat{u}_0, \hat{\zeta}, \hat{\varphi})$, and let (u_0, ζ, φ) be as in (i). We extend ζ and φ by zero outside the interval J_T and denote by u the corresponding solution. The latter exists up to some maximal time $T_*(u_0, \zeta, \varphi) > 0$ by the first part of the proposition. Let us show that $T < T_*(u_0, \zeta, \varphi)$, provided that $\delta > 0$ in (1.4) is sufficiently small. Indeed, the function $w = u - \hat{u}$ is a solution of the problem

$$\partial_t w - \nu\Delta(w + \xi) + f(w + \xi + \hat{u} + \hat{\zeta}) - f(\hat{u} + \hat{\zeta}) = \psi, \tag{1.6}$$

$$w(0, x) = w_0(x) \tag{1.7}$$

with $w_0 = u_0 - \hat{u}_0$, $\xi = \zeta - \hat{\zeta}$, and $\psi = \varphi - \hat{\varphi}$. For any $\mathcal{T} > 0$ and $F \in L^2(J_{\mathcal{T}}, H^{s-1})$, we have that

$$\Psi := e^{\nu t\Delta} w_0 + \int_0^t e^{\nu(t-\tau)\Delta} F \, \mathrm{d}\tau$$

belongs to $\mathcal{X}_{\mathcal{T},s}$ and

$$\|\Psi\|_{\mathcal{X}_{\mathcal{T},s}} \le C_1 \left(\|w_0\|_s + \|F\|_{L^2(J_{\mathcal{T}}, H^{s-1})}\right), \tag{1.8}$$

where C_1 does not depend on $\mathcal{T}$. Let us denote

$$\mathcal{T} = \sup\{t < T_*(\hat{u}_0, \hat{\zeta}, \hat{\varphi}) \wedge T_*(u_0, \zeta, \varphi) : \|w(t)\|_s < 1\}.$$

We will show that $\mathcal{T} > T$, provided that δ in (1.4) is sufficiently small. To this end, we apply (1.8) with

$$F := \nu\Delta\xi - f(w + \xi + \hat{u} + \hat{\zeta}) + f(\hat{u} + \hat{\zeta}) + \psi,$$

and note that $w = \Psi$ for this choice of F. Then

$$\begin{aligned} \|w\|_{\mathcal{X}_{\mathcal{T},s}} \le C_1\big(&\|w_0\|_s + \nu\|\xi\|_{L^2(J_{\mathcal{T}},H^{s+1})} + \|\psi\|_{L^2(J_{\mathcal{T}},H^{s-1})} \\ &+ \|f(w + \xi + \hat{u} + \hat{\zeta}) - f(\hat{u} + \hat{\zeta})\|_{L^2(J_{\mathcal{T}},H^{s-1})}\big). \end{aligned} \tag{1.9}$$

To estimate the term with f, we start with the polynomial part:

$$\begin{aligned} &\|P_p(w + \xi + \hat{u} + \hat{\zeta}) - P_p(\hat{u} + \hat{\zeta})\|_s \\ &\qquad \le C_2\, \|w + \xi\|_s \Big(\|w\|_s + \|\xi\|_s + \|\hat{u}\|_s + \|\hat{\zeta}\|_s + 1\Big)^{p-1}, \end{aligned} \tag{1.10}$$

where we used the fact that H^s is an algebra for $s > d/2$ and

$$\|ab\|_s \le C_3\, \|a\|_s \|b\|_s, \quad a, b \in H^s. \tag{1.11}$$

Then we write

$$g(w + \xi + \hat{u} + \hat{\zeta}) - g(\hat{u} + \hat{\zeta}) = (w + \xi) \int_0^1 g'(\tau(w + \xi) + \hat{u} + \hat{\zeta})\, \mathrm{d}\tau,$$

and apply inequality (1.11):

$$\|g(w + \xi + \hat{u} + \hat{\zeta}) - g(\hat{u} + \hat{\zeta})\|_s \le \|w + \xi\|_s \int_0^1 \|g'(\tau(w + \xi) + \hat{u} + \hat{\zeta})\|_s\, \mathrm{d}\tau. \tag{1.12}$$

Now we use the inequality

$$\|g'(a)\|_s \le C_4\, \|a\|_s^s, \quad a \in H^s, \tag{1.13}$$

which is obtained by applying the Sobolev inclusion $H^s \subset L^\infty$ and the Gagliardo–Nirenberg inequality

$$\|D^k a\|_{L^{2s/k}} \le C_5\, \|D^s a\|_{L^2}^{k/s} \|a\|_\infty^{1-k/s}$$

to the terms that arise in estimating the L^2 norms of the derivatives of $g'(u)$ (see Section 2 in [A76]), where $D^k a$ is the derivative of order k of the function a. Combining (1.12) and (1.13), we get

$$\|g(w + \xi + \hat{u} + \hat{\zeta}) - g(\hat{u} + \hat{\zeta})\|_s \le C_6\, \|w + \xi\|_s \Big(\|w\|_s + \|\xi\|_s + \|\hat{u}\|_s + \|\hat{\zeta}\|_s + 1\Big)^s.$$

This inequality, together with (1.4), (1.10), and the Young inequality, implies that

$$\|f(w+\xi+\hat{u}+\hat{\zeta})-f(\hat{u}+\hat{\zeta})\|_s \le C_7\left(\delta+\|w\|_s^m\right), \quad t<\mathcal{T}, \tag{1.14}$$

where $m = p \wedge (s+1)$ and $C_7 = C_7(\Lambda)$. Going back to (1.9), we obtain

$$\|w(t)\|^2_{\mathcal{X}_{\mathcal{T},s}} \le C_8\left(\delta + \int_0^t \|w(\tau)\|_s^{2m}\,\mathrm{d}\tau\right), \quad t<\mathcal{T}\wedge T. \tag{1.15}$$

Let us denote

$$\Phi(t) := \delta + \int_0^t \|w(\tau)\|_s^{2m}\,\mathrm{d}\tau.$$

Inequality (1.15) implies that

$$(\dot{\Phi})^{1/m} \le C_8\,\Phi,$$

which is equivalent to

$$\frac{\dot{\Phi}}{\Phi^m} \le C_8^m.$$

Integrating the latter, we get

$$\Phi(t) \le \begin{cases} \delta\exp(C_8^m t), & \text{if } m=1,\\ \delta(1-(m-1)C_8^m\delta^{m-1}t)^{-1/(m-1)}, & \text{if } m\ge 2,\end{cases} \qquad t<\mathcal{T}\wedge T.$$

Choosing δ sufficiently small, we see that

$$\Phi(t) < C_9\delta < 1, \quad t<\mathcal{T}\wedge T, \tag{1.16}$$

which implies that $\mathcal{T} > T$ and proves (i).

Step 2. Proof of (ii). Using inequalities (1.9), (1.14), (1.15), and (1.16), we get

$$\|w\|^2_{\mathcal{X}_{t,s}} \le C_{10}\left(\|w_0\|_s^2 + \|\xi\|^2_{L^2(J_T,H^{s+1})} + \|\psi\|^2_{L^2(J_T,H^{s-1})} + \int_0^t \|w\|^2_{\mathcal{X}_{\tau,s}}\mathrm{d}\tau\right)$$

for any $t\in J_T$. Applying the Gronwall inequality, we obtain (1.5) and complete the proof of the proposition. □

2 Main result

Let us take any $s>d/2$, $T>0$, $h\in L^2(J_T,H^{s-1})$, and $u_0\in H^s$, and consider problem (0.1), (0.3). The function η will be the control taking values in a finite-dimensional subspace $\mathcal{H}$ of H^{s+2} that will be specified below. Let us set

$$\Theta(u_0,h,T) := \{\eta\in L^2(J_T,H^{s-1}) : \text{s.t. (0.1), (0.3) has a solution in } \mathcal{X}_{T,s}\},$$

and recall that $\mathcal{R}(\cdot,\cdot,\cdot)$ is the resolving operator for (1.1), (0.3) and $\mathcal{R}_t(\cdot,\cdot,\cdot)$ is its evaluation at time t.

Definition 2.1. We shall say that Eq. (0.1) is approximately controllable by $\mathcal{H}$-valued control if for any $\varepsilon > 0$ and any $u_0, u_1 \in H^s$ there is a control $\eta \in \Theta(u_0, h, T) \cap L^2(J_T, \mathcal{H})$ such that

$$\|\mathcal{R}_T(u_0, 0, h + \eta) - u_1\|_s < \varepsilon.$$

To simplify the presentation, we assume that the leading coefficient c of the polynomial P_p in (0.2) equals to one. Indeed, the general case can be reduced to this one by a time scaling $\tau = c^{-1}t$ and noting that the below results do not change if we multiply ν, h and g by a constant.

Recall that $p \geq 2$ is the integer in (0.2). For any finite-dimensional subspace $\mathcal{H} \subset H^{s+2}$, let $\mathcal{C}(\mathcal{H})$ be the (nonconvex) cone defined by

$$\left\{ \eta - \sum_{m=1}^{n} \zeta_m^p : \ n \geq 1, \ \eta, \zeta_1, \ldots, \zeta_n \in \mathcal{H} \right\}.$$

We denote by $\mathcal{F}(\mathcal{H})$ the largest vector space contained in $H^{s+2} \cap \overline{\mathcal{C}(\mathcal{H})}^{H^s}$, where $\overline{\mathcal{C}(\mathcal{H})}^{H^s}$ is the closure of $\mathcal{C}(\mathcal{H})$ in H^s. It is easy to check that $\mathcal{F}(\mathcal{H})$ is well defined and finite-dimensional. Iterating this, we construct a non-decreasing sequence of finite-dimensional subspaces:

$$\mathcal{H}_0 = \mathcal{H}, \quad \mathcal{H}_j = \mathcal{F}(\mathcal{H}_{j-1}), \quad j \geq 1, \tag{2.1}$$

and denote $\mathcal{H}_\infty = \bigcup_{j=1}^{\infty} \mathcal{H}_j$.

Definition 2.2. We say that $\mathcal{H}$ is saturating if $\mathcal{H}_\infty$ is dense in H^s.

Our definition of saturating subspace is close to the definitions used in the papers [AS06, Shi06]. The difference is that here the subspaces $\mathcal{H}_j$ are defined in terms of the approximations of the elements of the form (**??**) and not by the elements themselves. This point is important for the examples of saturating subspaces given in Section 4.

The following is a more general version of the Main Theorem stated in the Introduction.

Theorem 2.3. *Assume that one of the following two conditions hold: a) $g = 0$; or b) $p > s > d/2$ and $g : \mathbb{R} \to \mathbb{R}$ is an arbitrary bounded smooth function with bounded derivatives. If $\mathcal{H}$ is saturating, then Eq. (0.1) is approximately controllable by $\mathcal{H}$-valued control.*

We derive this theorem from the following proposition proved in Section 3. Let us denote

$$\hat{\Theta}(u_0, h, T) := \{(\eta, \zeta) \in L^2(J_T, H^{s-1}) \times C(J_T, H^{s+1}) : \text{s.t. (1.1), (0.3)} \\ \text{has a solution in } \mathcal{X}_{T,s} \text{ with } \varphi = h + \eta\}.$$

Proposition 2.4. *Under the conditions of Theorem 2.3, for any $u_0, \eta \in H^{s+1}$, $\zeta \in H^{s+2}$, and $h \in L^2(J_T, H^{s-1})$, there is $\delta_0 > 0$ such that $(\delta^{-1/p}\eta, \delta^{-1}\zeta) \in$*

$\hat{\Theta}(u_0, h, \delta)$ *for any* $\delta \in (0, \delta_0)$*, and the following limit holds for the corresponding solution at time* $t = \delta$*:*

$$\mathcal{R}_\delta(u_0, \delta^{-1/p}\zeta, h + \delta^{-1}\eta) \to u_0 + \eta - \zeta^p \quad \textit{in } H^s \textit{ as } \delta \to 0^+.$$

The proof of this proposition will be given in Section 3.

Proof of Theorem 2.3. As discussed in the Introduction, the idea is to establish approximate controllability in small time to the points of the affine space $u_0 + \mathcal{H}_N$ by combining Proposition 2.4 and an induction argument in N. Then the saturation property will imply approximate controllability in small time to any point of H^s. Finally, controllability in any time T is proved by steering the system close to the target u_1 in small time, then forcing it to remain close to u_1 for a sufficiently long time. The accurate proof is divided into four steps.

Step 1. Controllability in small time to $u_0 + \mathcal{H}_0$*.* Let us assume for the moment that $u_0 \in H^{s+1}$. First we prove that problem (0.1), (0.3) is approximately controllable to the set $u_0 + \mathcal{H}_0$ in small time. More precisely, we show that, for any $\varepsilon > 0$, $\eta \in \mathcal{H}_0$, and $T_0 > 0$, there is a time $T < T_0$ and a control $\hat{\eta} \in \Theta(u_0, h, T) \cap L^2(J_T, \mathcal{H})$ such that

$$\|\mathcal{R}_T(u_0, 0, h + \hat{\eta}) - u_0 - \eta\|_s < \varepsilon.$$

Indeed, applying Proposition 2.4 for the couple $(\eta, 0)$, we see that

$$\mathcal{R}_\delta(u_0, 0, h + \delta^{-1}\eta) \to u_0 + \eta \quad \text{in } H^s \text{ as } \delta \to 0^+,$$

which gives the required result with $\hat{\eta} = \delta^{-1}\eta$ and $T = \delta$.

Step 2. Controllability in small time to $u_0 + \mathcal{H}_N$*.* We argue by induction. Assume that the approximate controllability of problem (0.1), (0.3) to the set $u_0 + \mathcal{H}_{N-1}$ is already proved. Let $\eta_1 \in \mathcal{H}_N$ be of the form

$$\eta_1 = \eta - \sum_{m=1}^{n} \zeta_m^p \tag{2.2}$$

for some integer $n \geq 1$ and vectors $\eta, \zeta_1, \ldots, \zeta_n \in \mathcal{H}_{N-1}$. Applying Proposition 2.4 for the couple $(0, \zeta_1)$, we see that

$$\mathcal{R}_\delta(u_0, \delta^{-1/p}\zeta_1, h) \to u_0 - \zeta_1^p \quad \text{in } H^s \text{ as } \delta \to 0. \tag{2.3}$$

By the uniqueness of the solution of the Cauchy problem, the following equality holds

$$\mathcal{R}_t(u_0 + \delta^{-1/p}\zeta_1, 0, h) = \mathcal{R}_t(u_0, \delta^{-1/p}\zeta_1, h) + \delta^{-1/p}\zeta_1, \quad t \in J_\delta.$$

Taking $t = \delta$ in this equality and using limit (2.3), we obtain

$$\|\mathcal{R}_\delta(u_0 + \delta^{-1/p}\zeta_1, 0, h) - u_0 + \zeta_1^p - \delta^{-1/p}\zeta_1\|_s \to 0 \quad \text{as } \delta \to 0.$$

Combining this with the fact that $\eta, \zeta_1 \in \mathcal{H}_{N-1}$, the induction hypothesis, and Proposition 1.1, we find a small time $T > 0$ and a control $\hat{\eta}_1 \in \Theta(u_0, h, T) \cap L^2(J_T, \mathcal{H})$ such that

$$\|\mathcal{R}_T(u_0, 0, h + \hat{\eta}_1) - u_0 - \eta + \zeta_1^p\|_s < \varepsilon.$$

Iterating this argument successively for the vectors $\zeta_2, \ldots, \zeta_n$, we construct a small time $\hat{T} > 0$ and a control $\hat{\eta} \in \Theta(u_0, h, \hat{T}) \cap L^2(J_{\hat{T}}, \mathcal{H})$ satisfying

$$\|\mathcal{R}_{\hat{T}}(u_0, 0, h + \hat{\eta}) - u_0 - \eta + \zeta_1^p + \ldots + \zeta_n^p\|_s = \|\mathcal{R}_{\hat{T}}(u_0, 0, h + \hat{\eta}) - u_0 - \eta_1\|_s < \varepsilon,$$

where we used (2.2). This proves the approximate controllability in small time to any point in $u_0 + \mathcal{H}_N$.

Step 3. Global approximate controllability in small time. Now let $u_1 \in H^s$ be arbitrary. As $\mathcal{H}_\infty$ is dense in H^s, there is an integer $N \geq 1$ and point $\hat{u}_1 \in u_0 + \mathcal{H}_N$ such that

$$\|u_1 - \hat{u}_1\|_s < \varepsilon/2. \tag{2.4}$$

By the results of Steps 1 and 2, for any $\varepsilon > 0$ and $T_0 > 0$ there is a time $T < T_0$ and a control $\hat{\eta} \in \Theta(u_0, h, T) \cap L^2(J_T, \mathcal{H})$ satisfying

$$\|\mathcal{R}_T(u_0, 0, h + \hat{\eta}) - u_1\|_s < \varepsilon/2.$$

Combining this with (2.4), we get approximate controllability in small time to u_1 from $u_0 \in H^{s+1}$. Taking control equal to zero on a small time interval and using the regularising property of the equation, we conclude small time approximate controllability starting from arbitrary $u_0 \in H^s$. By regularising property we mean that the solution becomes smooth at any time $t > 0$ when the initial point u_0 is in H^s. This can be seen, for example, by using the Duhamel formula and the regularising property of the heat semigroup.

Step 4. Global approximate controllability in fixed time T. Since we have global controllability in small time, to complete the proof of the theorem, it suffices to show that, for any $\varepsilon, T > 0$ and any $u_1 \in H^s$, there is a control $\eta \in \Theta(u_1, h, T) \cap L^2(J_T, \mathcal{H})$ such that

$$\|\mathcal{R}_T(u_1, 0, h + \eta) - u_1\|_s < \varepsilon. \tag{2.5}$$

Note that here the initial condition and the target coincide with u_1. It is not clear, whether it is possible or not to find a control taking values in $\mathcal{H}$ such that the solution starting from u_1 remains close to that point on all the time interval J_T. However, the argument below allows to have (2.5) precisely at time T.

By Proposition 1.1, there are numbers $r \in (0, \varepsilon)$ and $\tau > 0$ such that, for any $v \in B_{H^s}(u_1, r)$, we have $(0, 0) \in \hat{\Theta}(v, h, \tau)$ and

$$\|\mathcal{R}_t(v, 0, h) - u_1\|_s < \varepsilon \quad \text{for} \quad t \in J_\tau.$$

Thus starting from any point v in the r-neighborhood of u_1, we are guaranteed to remain ε-close to u_1 on the (uniform) time interval $[0, \tau]$. If $\tau > T$, then the proof is complete. Otherwise, applying the result of Step 3 with initial condition $u_0 = \mathcal{R}_\tau(v, h)$, small time $T' < T - \tau$, and target u_1, we find a control $\hat{\eta} \in \Theta(u_0, h, T') \cap L^2(J_{T'}, \mathcal{H})$ such that

$$\|\mathcal{R}_{T'}(u_0, 0, h + \hat{\eta}) - u_1\|_s < r.$$

Applying again Proposition 1.1, we conclude that, if $2\tau + T' > T$, then the proof is complete. Otherwise, we apply again the small time controllability property to return to the ball $B_{H^s}(u_1, r)$. After a finite number (less than the integer part of $T/\tau + 1$) of iterations, we complete the proof of the theorem. □

Now let us assume that the nonlinear term f in Eq. (0.1) is a polynomial of degree $p \geq 2$, i.e., $g = 0$. Recall that the space $\mathcal{H}(\mathcal{I})$ is defined by (0.4) for a finite symmetric set $\mathcal{I} \subset \mathbb{Z}^d$ containing the origin. Let us denote by $\tilde{\mathcal{I}}$ the set of all linear combinations of elements of $\mathcal{I}$ with integer coefficients. By definition, $\mathcal{I}$ is a generator if $\tilde{\mathcal{I}} = \mathbb{Z}^d$. Let $H^s(\mathcal{I})$ be the closure in H^s of the set

$$\operatorname{span}\{\sin\langle x, k\rangle, \cos\langle x, k\rangle : k \in \tilde{\mathcal{I}}\}.$$

Proposition 2.5. *The space $\mathcal{H}(\mathcal{I})$ is saturating if and only if $\mathcal{I}$ is a generator.*

This proposition is established in Section 4. We have the following more detailed version of Theorem 2.3.

Theorem 2.6. *Let $h \in L^2(J_T, H^{s-1}(\mathcal{I}))$ and let the function f be a polynomial of degree $p \geq 2$. Then Eq. (0.1) is approximately controllable if and only if $\mathcal{I}$ is a generator.*

Proof. By Proposition 2.5, if $\mathcal{I}$ is a generator, then $\mathcal{H}(\mathcal{I})$ is saturating. Thus Eq. (0.1) is approximately controllable by Theorem 2.3.

Now assume that $\mathcal{I}$ is not a generator. Then there is a vector $m \in \mathbb{Z}^d$ which does not belong to $\tilde{\mathcal{I}}$. The set of attainability from the origin defined by

$$\mathcal{A} := \{\mathcal{R}_T(0, 0, h + \eta) : \eta \in \Theta(u_0, h, T) \cap L^2(J_T, \mathcal{H})\}$$

is contained in $H^s(\mathcal{I})$. Indeed, this follows from the assumption that $h \in L^2(J_T, H^{s-1}(\mathcal{I}))$, the fact that the space $H^s(\mathcal{I})$ is invariant for the linear dynamics of the heat equation, and the assumption that the nonlinear term f in Eq. (0.1) is a polynomial (so f maps $H^s(\mathcal{I})$ to itself). Thus the functions $\cos\langle x, m\rangle$ and $\sin\langle x, m\rangle$ will be orthogonal to $\mathcal{A}$, hence $\mathcal{A}$ is not dense in H^s. This proves that Eq. (0.1) is not approximately controllable by $\mathcal{H}(\mathcal{I})$-valued control. □

3 Proof of Proposition 2.4

Assume that $u_0, \eta \in H^{s+1}$, $\zeta \in H^{s+2}$, and $h \in L^2(J_T, H^{s-1})$ are such that

$$\|u_0\|_{s+1} + \|\eta\|_{s+1} + \|\zeta\|_{s+2} + \|h\|_{L^2(J_T, H^{s-1})} < R. \tag{3.1}$$

Let us take any $\delta > 0$ and consider the equation

$$\partial_t u - \nu\Delta(u + \delta^{-1/p}\zeta) + f(u + \delta^{-1/p}\zeta) = h + \delta^{-1}\eta. \tag{3.2}$$

By Proposition 1.1, problem (3.2), (0.3) has a unique maximal solution defined on an interval $[0, T_*)$, where $T_* := T_*(u_0, \delta^{-1/p}\zeta, h + \delta^{-1}\eta) > 0$. Moreover,

$$\|u(t)\|_s \to +\infty \quad \text{as } t \to T_*^-, \tag{3.3}$$

when $T_* < \infty$. We will show that

(1) there is a number $\delta_0 > 0$ such that for any $\delta < \delta_0$, the solution of Eq. (3.2) is defined for $t \in [0, T_*)$ with $T_* > \delta$;

(2) the following limit holds

$$u(\delta) \to u_0 + \eta - \zeta^p \quad \text{in } H^s \text{ as } \delta \to 0^+, \tag{3.4}$$

uniformly with respect to u_0, η, ζ and h satisfying (3.1).

Inspired by some ideas from [JK85, AS06], we make a time substitution and consider the functions

$$\begin{aligned} w(t) &:= u_0 + t(\eta - \zeta^p), \\ v(t) &:= u(\delta t) - w(t), \end{aligned} \tag{3.5}$$

which are well defined for $t < \delta^{-1}T_*$. Then v is a solution of problem

$$\partial_t v - \nu\delta\Delta(v + w + \delta^{-1/p}\zeta) + \delta f(v + w + \delta^{-1/p}\zeta) - \zeta^p = \delta h, \tag{3.6}$$

$$v(0) = 0. \tag{3.7}$$

Taking the scalar product in L^2 of Eq. (3.6) with $(-\Delta)^s v + v$ and using the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} \partial_t\|v\|_s^2 + \nu\delta\|\nabla v\|_s^2 \le{}& C_1\delta\left(\|w\|_{s+1} + \|h\|_{s-1}\right)\|\nabla v\|_s \\ &+ C_1\left(\delta^{1-1/p}\|\zeta\|_{s+2} + \|\delta f(v + w + \delta^{-1/p}\zeta) - \zeta^p\|_s\right)\|v\|_s. \end{aligned} \tag{3.8}$$

From the Young inequality and (3.1) we derive

$$\begin{aligned} C_1\delta\left(\|w\|_{s+1} + \|h\|_{s-1}\right)\|\nabla v\|_s &\le \frac{\nu\delta}{2}\|\nabla v\|_s^2 + C_2\delta\left(\|w\|_{s+1}^2 + \|h\|_{s-1}^2\right) \\ &\le \frac{\nu\delta}{2}\|\nabla v\|_s^2 + C_3\delta\left(1 + \|h\|_{s-1}^2\right), \end{aligned} \tag{3.9}$$

for $t \le 1 \wedge (\delta^{-1}T_*)$, where the constant C_3 (and almost all the constants C_i below) depends on R. The nonlinear term is estimated as in Section 1. We assume that $\delta \le 1$ and estimate the polynomial part as follows:

$$\|\delta P_p(v + w + \delta^{-1/p}\zeta) - \zeta^p\|_s \le C_4\delta^{1/p}\left(\|v\|_s^p + 1\right) \tag{3.10}$$

for $t \le 1 \wedge (\delta^{-1}T_*)$, where we used (1.11), (3.1), and (3.5). For the term g, we use the inequality (cf. (1.13))

$$\|g(a)\|_s \le C_5 \|a\|_s^s, \quad a \in H^s,$$

and the assumptions that $p > s$, $\delta \le 1$, and (3.1) to prove that:

$$\|\delta g(v + w + \delta^{-1/p}\zeta)\|_s \le C_6 \delta^{1/p} \left(\|v\|_s^s + 1\right) \tag{3.11}$$

for $t \le 1 \wedge (\delta^{-1}T_*)$, Combining inequalities (3.8)-(3.11) and the assumption that $p \ge 2$, we arrive at

$$\partial_t \|v\|_s^2 \le C_7 \delta^{1/p} \left(\|h\|_{s-1}^2 + \|v\|_s^{p+1} + 1\right) = C_7 \delta^{1/p} \left(\psi + \|v\|_s^{p+1}\right), \tag{3.12}$$

where $\psi := \|h\|_{s-1}^2 + 1$ and the constant $C_7 > 0$ does not depend on δ. Let us set

$$\Phi(t) := A + C_7 \delta^{1/p} \int_0^t \|v\|_s^{p+1} \, \mathrm{d}\tau, \tag{3.13}$$

where

$$A := C_7 \delta^{1/p} \int_0^1 \psi \, \mathrm{d}\tau.$$

Inequality (3.12) and initial condition (3.7) imply that

$$(\dot{\Phi})^{2/(p+1)} \le (C_7 \delta^{1/p})^{2/(p+1)} \Phi.$$

The latter is equivalent to

$$\frac{\dot{\Phi}}{\Phi^{(p+1)/2}} \le C_7 \delta^{1/p}.$$

Integrating this inequality, we obtain

$$\Phi(t) \le A \left(1 - \frac{(p-1)}{2} C_7 \delta^{1/p} A^{(p-1)/2} t\right)^{-2/(p-1)}, \quad t < 1 \wedge (\delta^{-1}T_*) \wedge T_1,$$

where

$$\begin{aligned} T_1 &:= \left((p-1) C_7 \delta^{1/p} A^{(p-1)/2}\right)^{-1} \\ &= \left((p-1) C_7^{(p+1)/2} \delta^{(p+1)/(2p)} \left(\int_0^1 \psi \, \mathrm{d}\tau\right)^{(p-1)/2}\right)^{-1}. \end{aligned}$$

We choose $\delta_0 \in (0,1)$ so small that $T_1 \ge 1$ for any $\delta < \delta_0$ and

$$\Phi(t) \le 2A = 2C_7 \delta^{1/p} \int_0^1 \psi \, \mathrm{d}\tau, \quad t < 1 \wedge (\delta^{-1}T_*). \tag{3.14}$$

From this and (3.3) we derive that

$$\delta^{-1}T_* > 1 \quad \text{for } \delta < \delta_0,$$

which yields (1). Combining (3.12)-(3.14), we see that

$$\|v(1)\|_s \le C_8 \delta^{1/p},$$

so $v(1) \to 0$ in H^s as $\delta \to 0^+$. This gives (3.4) and completes the proof of the proposition.

4 Saturating subspaces

Let $\mathcal{H}(\mathcal{I})$ be the space defined by (0.4) and $\mathcal{I} \subset \mathbb{Z}^d$ be a finite symmetric set containing the origin. Here we prove Proposition 2.5.

Proof of Proposition 2.5. Step 1. Sufficiency of the condition. Assume that $\mathcal{I}$ is a generator and $\{\mathcal{H}_k(\mathcal{I})\}$ and $\mathcal{H}_\infty(\mathcal{I})$ are the vector spaces defined by (2.1) with $\mathcal{H} = \mathcal{H}(\mathcal{I})$. We distinguish two cases.

Case 1. p is odd. This case is particularly simple due to the following representation of the space $\mathcal{F}(\mathcal{H})$.

Lemma 4.1. *If p is odd, then*

$$\begin{aligned} \mathcal{F}(\mathcal{H}) &= \operatorname{span}\{\mathcal{H}, \{\zeta^p : \ \zeta \in \mathcal{H}\}\} && (4.1)\\ &= \operatorname{span}\{\mathcal{H}, \{\zeta_1 \cdot \ldots \cdot \zeta_p : \ \zeta_i \in \mathcal{H}, \ i = 1, \ldots, p\}\}. && (4.2) \end{aligned}$$

This lemma is proved at the end of this section. Here we apply it to show that

$$\cos\langle x, l \pm m\rangle, \ \sin\langle x, l \pm m\rangle \in \mathcal{H}_1(\mathcal{I}) \quad \text{for } l, m \in \mathcal{I}. \tag{4.3}$$

Indeed, this easily follows from (4.2) by taking $\zeta_1 = \ldots = \zeta_{p-2} = 1$ and choosing appropriately ζ_{p-1} and ζ_p from the identities

$$\begin{aligned} \cos\langle x, l \pm m\rangle &= \cos\langle x, l\rangle \cos\langle x, m\rangle \mp \sin\langle x, l\rangle \sin\langle x, m\rangle,\\ \sin\langle x, l \pm m\rangle &= \sin\langle x, l\rangle \cos\langle x, m\rangle \pm \cos\langle x, l\rangle \sin\langle x, m\rangle. \end{aligned}$$

Combining (4.3) with the fact that $\mathcal{I}$ is a generator, we see that

$$\mathcal{H}_\infty(\mathcal{I}) = \operatorname{span}\{\cos\langle x, m\rangle, \sin\langle x, m\rangle : \ m \in \mathbb{Z}^d\}.$$

Thus $\mathcal{H}_\infty(\mathcal{I})$ is dense in H^s and $\mathcal{H}(\mathcal{I})$ is saturating.

Case 2. p is even. Here we show that

$$\cos\langle x, l \pm m\rangle, \sin\langle x, l \pm m\rangle \in \mathcal{H}_2(\mathcal{I}) \quad \text{for } l, m \in \mathcal{I}. \tag{4.4}$$

We first assume that $p \geq 4$. Let us check that

$$\cos\langle x, 2m\rangle \in \mathcal{H}_1(\mathcal{I}) \quad \text{for } m \in \mathcal{I}. \tag{4.5}$$

Indeed, for any $\varepsilon > 0$ and $\alpha = -2/(p-2)$, we have the equalities

$$(\varepsilon^\alpha + \varepsilon \cos\langle x, m\rangle)^p = \frac{p(p-1)}{4}(1 + \cos\langle x, 2m\rangle) + \varepsilon^{\alpha p} + p\varepsilon^{\alpha(p-1)+1}\cos\langle x, m\rangle + a(\varepsilon),$$

$$(\varepsilon^\alpha + \varepsilon \sin\langle x, m\rangle)^p = \frac{p(p-1)}{4}(1 - \cos\langle x, 2m\rangle) + \varepsilon^{\alpha p} + p\varepsilon^{\alpha(p-1)+1}\sin\langle x, m\rangle + b(\varepsilon),$$

where $a(\varepsilon)$, $b(\varepsilon) \to 0$ in H^s as $\varepsilon \to 0$. As

$$1, \cos\langle x, m\rangle, \sin\langle x, m\rangle \in \mathcal{H}(\mathcal{I}),$$

we obtain (4.5). Now we use a similar argument to prove (4.4). The fact that

$$\cos\langle x, l+m\rangle \in \mathcal{H}_2(\mathcal{I}) \quad \text{for } l, m \in \mathcal{I} \tag{4.6}$$

is checked by using the equalities

$$(\varepsilon^\alpha + \varepsilon(\cos\langle x,l\rangle \pm \cos\langle x,m\rangle))^p = \pm p(p-1)\cos\langle x,l\rangle\cos\langle x,m\rangle + \eta^c_\pm(\varepsilon) + a_\pm(\varepsilon),$$
$$(\varepsilon^\alpha + \varepsilon(\sin\langle x,l\rangle \pm \sin\langle x,m\rangle))^p = \pm p(p-1)\sin\langle x,l\rangle\sin\langle x,m\rangle + \eta^s_\pm(\varepsilon) + b_\pm(\varepsilon),$$

where $\eta^c_\pm(\varepsilon), \eta^s_\pm(\varepsilon) \in \mathcal{H}_1(\mathcal{I})$ (here we use (4.5)) and $a_\pm(\varepsilon), b_\pm(\varepsilon) \to 0$ in H^s as $\varepsilon \to 0$. The remaining assertions in (4.4) are proved in a similar way. From (4.4) and the fact that $\mathcal{I}$ is a generator we derive

$$\mathcal{H}_\infty(\mathcal{I}) \supset \operatorname{span}\{\cos\langle x,l\rangle, \sin\langle x,l\rangle :\ l \in \mathbb{Z}^d\},$$

which proves that $\mathcal{H}(\mathcal{I})$ is saturating when $p \geq 4$.

The case $p = 2$ is easier. To show (4.5), we use the equalities

$$\cos^2\langle x,m\rangle = \frac{1}{2}(1 + \cos\langle x,2m\rangle), \quad \sin^2\langle x,m\rangle = \frac{1}{2}(1 - \cos\langle x,2m\rangle)$$

and the assumption that $1 \in \mathcal{H}(\mathcal{I})$. It follows that

$$\cos^2\langle x,m\rangle, \sin^2\langle x,m\rangle \in \mathcal{H}_1(\mathcal{I}) \quad \text{for } m \in \mathcal{I}.$$

Then (4.6) follows from the equalities

$$(\cos\langle x,l\rangle \pm \cos\langle x,m\rangle)^2 = \cos^2\langle x,l\rangle + \cos^2\langle x,m\rangle \pm 2\cos\langle x,l\rangle\cos\langle x,m\rangle,$$
$$(\sin\langle x,l\rangle \pm \sin\langle x,m\rangle)^2 = \sin^2\langle x,l\rangle + \sin^2\langle x,m\rangle \pm 2\sin\langle x,l\rangle\sin\langle x,m\rangle.$$

The proof of the other assertions in (4.4) is similar. As in the case $p \geq 4$, we conclude that $\mathcal{H}(\mathcal{I})$ is saturating.

Step 2. Necessity of the condition. Now assume that $\mathcal{I}$ is not a generator. Then there is a vector $m \in \mathbb{Z}^d$ which does not belong to the set $\tilde{\mathcal{I}}$ of all linear combinations of elements of $\mathcal{I}$ with integer coefficients. It is easy to see that

$$\mathcal{H}_\infty(\mathcal{I}) \subset \operatorname{span}\{\cos\langle x,l\rangle, \sin\langle x,l\rangle :\ l \in \tilde{\mathcal{I}}\}.$$

Thus the functions $\cos\langle x,m\rangle$ and $\sin\langle x,m\rangle$ are orthogonal to $\mathcal{H}_\infty(\mathcal{I})$. This implies that $\mathcal{H}_\infty(\mathcal{I})$ is not dense in H^s, so $\mathcal{H}(\mathcal{I})$ is not saturating. □

The simplest example of saturating space of form (0.4) will be the $(2d+1)$-dimensional space corresponding to the set

$$\mathcal{I} = \{0, \pm e_i :\ i = 1, \dots, d\} \subset \mathbb{Z}^d,$$

where $\{e_j\}_{j=1}^d$ is the standard basis in $\mathbb{R}^d$. The following result, combined with Proposition 2.5, gives a simple way for constructing more saturating spaces.

Theorem 4.2. *A set $\mathcal{I} \subset \mathbb{Z}^d$ is a generator if and only if the greatest common divisor of the set*

$$\{\det(a_1, \ldots, a_d) : a_i \in \mathcal{I},\, i = 1, \ldots, d\}$$

is 1*, where* $\det(a_1, \ldots, a_d)$ *is the determinant of the* $d \times d$ *matrix with columns* $a_1, \ldots, a_d$.

See Section 3.7 in [Jac85] for the proof of this theorem.

Proof of Lemma 4.1. Equality (4.1) follows immediately from the fact that p is odd. Let us denote by $\mathcal{G}_1$ and $\mathcal{G}_2$ the spaces on the right-hand sides of (4.1) and (4.2), respectively. Obviously, $\mathcal{G}_1 \subset \mathcal{G}_2$. To see the inclusion $\mathcal{G}_2 \subset \mathcal{G}_1$, let us take any $\zeta_1, \ldots, \zeta_p \in \mathcal{H}$, and consider the function

$$F : \mathbb{R}^p \to \mathcal{G}_1, \quad (x_1, \ldots, x_p) \mapsto (x_1\zeta_1 + \ldots + x_p\zeta_p)^p.$$

As $\mathcal{G}_1$ is closed, it contains the derivative

$$\frac{\partial^p}{\partial_{x_1} \ldots \partial_{x_p}} F(0, \ldots, 0) = p!\, \zeta_1 \cdot \ldots \cdot \zeta_p.$$

This implies that $\zeta_1 \cdot \ldots \cdot \zeta_p \in \mathcal{G}_1$, so $\mathcal{G}_2 \subset \mathcal{G}_1$. □

References

[A76] D. R. Adams. On the existence of capacitary strong type estimates in R^n. *Arkiv för Matematik*, 14(1):125–140, 1976.

[AS05] A. A. Agrachev and A. V. Sarychev. Navier–Stokes equations: controllability by means of low modes forcing. *J. Math. Fluid Mech.*, 7(1):108–152, 2005.

[AS06] A. A. Agrachev and A. V. Sarychev. Controllability of 2D Euler and Navier–Stokes equations by degenerate forcing. *Comm. Math. Phys.*, 265(3):673–697, 2006.

[AS08] A. A. Agrachev and A. V. Sarychev. Solid controllability in fluid dynamics. In *Instability in Models Connected with Fluid Flows. I*, volume 6 of *Int. Math. Ser. (N.Y.)*, pages 1–35. Springer, New York, 2008.

[BGN20] P.-M. Boulvard, P. Gao, and V. Nersesyan. Controllability and ergodicity of 3D primitive equations driven by a finite-dimensional force. *In preparation*, 2020.

[GHHM18] N. E. Glatt-Holtz, D. P. Herzog, and J.C. Mattingly. Scaling and saturation in infinite-dimensional control problems with applications to stochastic partial differential equations. *Ann. PDE*, 4(2):Art. 16, 103, 2018.

[Jac85] N. Jacobson. *Basic algebra. I.* W. H. Freeman and Company, New York, Second edition, 1985.

[JK85] V. Jurdjevic and I. Kupka. Polynomial control systems. *Math. Ann.*, 272(3):361–368, 1985.

[Jur97] V. Jurdjevic. *Geometric control theory*, volume 52 of *Cambridge Studies in Advanced Mathematics*. 1997.

[KNS19] S. Kuksin, V. Nersesyan, and A. Shirikyan. Mixing via controllability for randomly forced nonlinear dissipative PDEs. *J. Éc. Polytech. Math.*, 7:871–896, 2020.

[Ner10] H. Nersisyan. Controllability of 3D incompressible Euler equations by a finite-dimensional external force. *ESAIM Control Optim. Calc. Var.*, 16(3):677–694, 2010.

[Ner11] H. Nersisyan. Controllability of the 3D compressible Euler system. *Comm. Partial Differential Equations*, 36(9):1544–1564, 2011.

[Ner15] V. Nersesyan. Approximate controllability of Lagrangian trajectories of the 3D Navier–Stokes system by a finite-dimensional force. *Nonlinearity*, 28(3):825–848, 2015.

[PR18] D. Phan and S. S. Rodrigues. Approximate controllability for Navier–Stokes equations in 3D rectangles under Lions boundary conditions. *J. Dyn. Control Syst.*, 25(3):351–376, 2019.

[QS07] P. Quittner and P. Souplet. *Superlinear parabolic problems: blow-up, global existence and steady states*. Birkhäuser Verlag, Basel, 2007.

[Rod06] S. S. Rodrigues. Navier–Stokes equation on the rectangle: Controllability by means of low mode forcing. *J. Dyn. Control Syst.*, 12(4):517–562, 2006.

[Sar12] A. Sarychev. Controllability of the cubic Schrödinger equation via a low-dimensional source term. *Math. Control Relat. Fields*, 2(3):247–270, 2012.

[Shi06] A. Shirikyan. Approximate controllability of three-dimensional Navier–Stokes equations. *Comm. Math. Phys.*, 266(1):123–151, 2006.

[Shi07] A. Shirikyan. Exact controllability in projections for three-dimensional Navier–Stokes equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 24(4):521–537, 2007.

[Shi14] A. Shirikyan. Approximate controllability of the viscous Burgers equation on the real line. In *Geometric control theory and sub-Riemannian geometry*, volume 5, pages 351–370. Springer, Cham, 2014.

[Shi18] A. Shirikyan. Control theory for the Burgers equation: Agrachev-Sarychev approach. *Pure Appl. Funct. Anal.*, 3(1):219–240, 2018.