Communications in
**Mathematical
Physics**

Growth of Sobolev Norms and Controllability of the Schrödinger Equation

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Abstract: In this paper we obtain a stabilization result for both linear and nonlinear Schrödinger equations under generic assumptions on the potential. Then we consider the Schrödinger equations with a potential which has a random time-dependent amplitude. We show that if the distribution of the amplitude is sufficiently non-degenerate, then any trajectory of the system is almost surely non-bounded in Sobolev spaces.

Contents

1. Introduction

We consider the problem

$$i\dot{z} = -\Delta z + V(x)z + u(t)Q(x)z, \quad x \in D, \tag{1.1}$$

$$z|_{\partial D} = 0, \tag{1.2}$$

$$z(0, x) = z_0(x), \tag{1.3}$$

where $D \subset \mathbb{R}^m$ is a bounded domain with smooth boundary, $V, Q \in C^\infty(\overline{D}, \mathbb{R})$ are given functions, u is the control, and z is the state. Under some hypotheses on V and Q (see Condition 3.2), we prove a stabilization result for problem (1.1), (1.2). This result is then applied to show that almost any trajectory of a random Schrödinger equation is non-bounded in Sobolev spaces. As it is shown in Sect. 3.4, the hypotheses on V and Q are in some sense generic.

Let us recall some previous results on the controllability of Schrödinger equations. A general negative result for bilinear control systems is obtained by Ball, Marsden and Slemrod [5]. Application of this result to (1.1), (1.2) implies that the set of attainable points from any initial data in H^2 admits a dense complement in H^2. We refer the reader to the papers [1,3,4,27,29] and the references therein for controllability of finite-dimensional systems. In [7], Beauchard proves that one can obtain an exact controllability result if the phase space is properly chosen. More precisely, in the case $m = 1$, $V(x) = 0$ and $Q(x) = x$ exact controllability of the problem is proved in H^7-neighborhoods of the eigenstates. Beauchard and Coron [8] established later a partial global exact controllability result, showing that the system in question is also controlled between some neighborhoods of any two eigenstates. A stabilization property for finite-dimensional approximations of the Schrödinger equation is obtained by Beauchard et al., in [9], which was generalized by Beauchard and Mirrahimi [10] to the infinite-dimensional case for $m = 1$, $V(x) = 0$ and $Q(x) = x$ (see also the paper by Mirrahimi [22]). Recently Chambrion et al. [14], under some assumptions on $V, Q \in C^\infty(\overline{D}, \mathbb{R})$, derived the approximate controllability of (1.1), (1.2) in L^2 from the controllability of finite-dimensional projections. See also the papers [6,12,16,20,21,32] and the references therein for controllability results by boundary controls and controls supported in a given subdomain and the book [15] by Coron for an introduction to the later developments and methods in the control theory of nonlinear systems.

The main result of this paper states that any neighborhood of the first eigenfunction of the operator $-\Delta + V$ is attainable from any initial point $z_0 \in H^2$. This result, combined with the time reversibility property of the system and the fact that the equation is linear, implies an approximate controllability property in L^2.

Let us describe in a few words the main ideas of the proof. As V, Q and u are real-valued, the L^2 norm is preserved by the flow of the system. Thus it suffices to consider the restriction of (1.1), (1.2) to the unit sphere S in L^2. We introduce a Lyapunov function $\mathcal{V}(z)$ that controls the H^2-norm of z. The infimum of $\mathcal{V}$ on the sphere S is attained at the first eigenfunction $e_{1,V}$ of the operator $-\Delta + V$. Using the same idea as in [9], which consists in generating trajectories with Lyapunov techniques, we choose a feedback law $u(z)$ such that the function $\mathcal{V}$ decreases on the solutions of the corresponding system:

$$\mathcal{V}(\mathcal{U}_t(z_0, u)) < \mathcal{V}(z_0), \quad t > 0,$$

where $\mathcal{U}_t(\cdot, u)$ is the resolving operator of (1.1), (1.2). Then iterating this construction and using the fact that the system is autonomous, we prove that the H^2-weak ω-limit set of any solution contains the minimum point of the function $\mathcal{V}$, i.e. the eigenfunction $e_{1,V}$ (see Sects. 3.1 and 3.3). The ideas of the proof work also in the case of the nonlinear equation. Furthermore, these controllability results are generalized in [24] to higher Sobolev spaces H^l, $l > 2$. Under the same hypotheses on V and Q, we prove global approximate controllability for problem (1.1), (1.2).

We next use the above-mentioned controllability result to study the large time behavior of solutions of a random Schrödinger equation. We prove that if the distribution of the random potential is sufficiently non-degenerate (see Condition 4.6), then the trajectories of the system are almost surely non-bounded. It is interesting to compare this

result with that of Eliasson and Kuksin [17], where KAM-technique is applied to prove the reducibility of a linear Schrödinger equation with time-quasiperiodic potential. In particular, it is proved that for most values of the frequency vector the Sobolev norms of the solutions are bounded. Examples of unbounded solutions of 1D linear Schrödinger equations with some random potentials are constructed in [11,18], where also the growth rate estimates are given. Our assumptions on the distribution of the potential are more general, and the proof also works in the case of the nonlinear equation. However, at this level of generality, we do not have any lower bound on the rate of growth of Sobolev norms. Let us mention also the papers [30,31] by Wang, where the growth of Sobolev norms for linear Schrödinger equations are estimated and some time periodic potentials are constructed such that the Sobolev norms of the solutions for the corresponding problem remain bounded.

The idea of the proof is to show that the first entrance time to any ball centered at the origin in $H^{-\varepsilon}$ is almost surely finite. This implies immediately that almost any trajectory of the system approaches the origin arbitrarily closely in $H^{-\varepsilon}$. Combining this with the fact that the L^2-norm is preserved, we conclude that almost any trajectory is non-bounded in H^l for any $l > 0$.

In conclusion, let us note that the results of this paper imply the irreducibility in L^2 of the Markov chain associated with (1.1). This property is not sufficient to prove the ergodicity of the dynamics generated by the Schrödinger equation with a random potential. However, in the case of finite-dimensional approximations, that question is treated in the paper [23], in which an exponential mixing property is established. We hope the methods developed in this work will help to tackle the infinite-dimensional case.

Notation. In this paper we use the following notation. Let $D \subset \mathbb{R}^m$, $m \geq 1$ be a bounded domain with smooth boundary. Let $H^s := H^s(D)$ be the Sobolev space of order $s \in \mathbb{R}$ endowed with the norm $\| \cdot \|_s$. Consider the operators $-\Delta z + V z$, $z \in \mathcal{D}(-\Delta + V) := H_0^1 \cap H^2$, where $V \in C^\infty(\bar{D}, \mathbb{R})$. We denote by $\{\lambda_{j,V}\}$ and $\{e_{j,V}\}$ the sets of eigenvalues and normalized eigenfunctions of $-\Delta + V$. Let $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ be the scalar product and the norm in the space L^2. Let S be the unit sphere in L^2. For a Banach space X, we shall denote by $B_X(a, r)$ the open ball of radius $r > 0$ centered at $a \in X$.

2. Preliminaries

The following lemma establishes the well-posedness of system (1.1)–(1.3).

Lemma 2.1. *For any $z_0 \in H_0^1 \cap H^2$ (resp. $z_0 \in L^2$) and for any $u \in L_{loc}^1([0, \infty), \mathbb{R})$ problem (1.1)–(1.3) has a unique solution $z \in C([0, \infty), H^2)$ (resp. $z \in C([0, \infty), L^2)$). Furthermore, the resolving operator $\mathcal{U}_t(\cdot, u) : L^2 \to L^2$ taking z_0 to $z(t)$ satisfies the relation*

$$\|\mathcal{U}_t(z_0, u)\| = \|z_0\|, \quad t \geq 0. \tag{2.1}$$

See [13] for the proof. Notice that the conservation of L^2-norm implies that it suffices to consider the controllability properties of (1.1), (1.2) on the unit sphere S.

In Sect. 4.2, we replace the control u by a random process. Namely, we consider the equation

$$i\dot{z} = -\Delta z + V(x)z + \beta(t)Q(x)z, \quad x \in D,$$

where $\beta(t)$ is a random process of the form

$$\beta(t) = \sum_{k=0}^{+\infty} I_k(t)\eta_k(t-k), \quad t \geq 0. \tag{2.2}$$

Here $I_k(\cdot)$ is the indicator function of the interval $[k, k+1)$ and η_k are independent identically distributed (i.i.d.) random variables in $L^2([0,1], \mathbb{R})$.

Let z_0 be a L^2-valued random variable independent of $\{\eta_k\}$. Denote by $\mathcal{F}_k$ the σ-algebra generated by $z_0, \eta_0, \ldots, \eta_{k-1}$.

Lemma 2.2. *Under the above conditions, $\mathcal{U}_k(\cdot, \beta)$ is a homogeneous Markov chain with respect to $\mathcal{F}_k$.*

This lemma is proved by standard arguments (e.g., see [25]).

3. Controllability of the Schrödinger Equation

3.1. Stabilization result. Let us introduce the Lyapunov function

$$\mathcal{V}(z) := \alpha \|(-\Delta + V)P_{1,V}z\|^2 + 1 - |\langle z, e_{1,V}\rangle|^2, \quad z \in S \cap H_0^1 \cap H^2,$$

where $\alpha > 0$ and $P_{1,V}z := z - \langle z, e_{1,V}\rangle e_{1,V}$ is the orthogonal projection in L^2 onto the closure of the vector span of $\{e_{k,V}\}_{k\geq 2}$. Notice that $\mathcal{V}(z) \geq 0$ for all $z \in S \cap H_0^1 \cap H^2$ and $\mathcal{V}(z) = 0$ if and only if $z = ce_{1,V}, |c| = 1$. For any $z \in S \cap H_0^1 \cap H^2$, we have

$$\mathcal{V}(z) \geq \alpha \|(-\Delta + V)P_{1,V}z\|^2 \geq \frac{\alpha}{2}\|\Delta(P_{1,V}z)\|^2 - C_1 \geq \frac{\alpha}{4}\|\Delta z\|^2 - C_2.$$

Thus

$$C(1 + \mathcal{V}(z)) \geq \|z\|_2 \tag{3.1}$$

for some constant $C > 0$. Following the ideas of [9], we wish to choose a feedback law $u(\cdot)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}(z(t)) \leq 0$$

for the solution $z(t)$ of (1.1)–(1.3). Let us assume that $\Delta z(t) \in H_0^1 \cap H^2$ for all $t \geq 0$. Using (1.1), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}(z(t)) = 2\alpha\,\mathrm{Re}(\langle(-\Delta+V)P_{1,V}\dot{z}, (-\Delta+V)P_{1,V}z\rangle) - 2\,\mathrm{Re}(\langle\dot{z}, e_{1,V}\rangle\langle e_{1,V}, z\rangle)$$

$$= 2\alpha\,\mathrm{Re}(\langle(-\Delta+V)P_{1,V}(i\Delta z - iVz - iuQz), (-\Delta+V)P_{1,V}z\rangle)$$
$$\quad -2\,\mathrm{Re}(\langle i\Delta z - iVz - iuQz, e_{1,V}\rangle\langle e_{1,V}, z\rangle).$$

Integrating by parts and using the fact that

$$(-\Delta+V)P_{1,V}z|_{\partial D} = z|_{\partial D} = e_{1,V}|_{\partial D} = 0,$$

we obtain

$$2\alpha \operatorname{Re}(\langle -i(-\Delta + V)^2 P_{1,V} z, (-\Delta + V) P_{1,V} z \rangle) - 2 \operatorname{Re}(\langle i\Delta z - iVz, e_{1,V} \rangle \langle e_{1,V}, z \rangle)$$
$$= 2\alpha \operatorname{Re}(\langle -i\nabla(-\Delta + V) P_{1,V} z, \nabla(-\Delta + V) P_{1,V} z \rangle)$$
$$+ 2\alpha \operatorname{Re}(\langle -iV(-\Delta + V) P_{1,V} z, (-\Delta + V) P_{1,V} z \rangle)$$
$$+ 2\lambda_{1,V} \operatorname{Re}(\langle iz, e_{1,V} \rangle \langle e_{1,V}, z \rangle) = 0.$$

Thus

$$\frac{d}{dt} \mathcal{V}(z(t)) = 2u \operatorname{Im}(\alpha \langle (-\Delta + V) P_{1,V}(Qz), (-\Delta + V) P_{1,V} z \rangle - \langle Qz, e_{1,V} \rangle \langle e_{1,V}, z \rangle).$$

Let us take

$$u(z) := -\delta \operatorname{Im}(\langle \alpha(-\Delta + V) P_{1,V}(Qz), (-\Delta + V) P_{1,V} z \rangle - \langle Qz, e_{1,V} \rangle \langle e_{1,V}, z \rangle), \quad (3.2)$$

where $\delta > 0$ is a small constant. Then

$$\frac{d}{dt} \mathcal{V}(z(t)) = -\frac{2}{\delta} u^2(z(t)). \tag{3.3}$$

Consider the equation

$$i\dot{z} = -\Delta z + V(x)z + u(z)Q(x)z, \quad x \in D. \tag{3.4}$$

Proposition 3.1. *For any $z_0 \in H_0^1 \cap H^2$ problem* (3.4), (1.2), (1.3) *has a unique solution $z \in C([0, \infty), H_0^1 \cap H^2)$. Moreover, the following properties hold:*

(i) *If $\Delta z_0 \in H_0^1 \cap H^2$, then $\Delta z \in C([0, \infty), H_0^1 \cap H^2)$.*
(ii) *Let $\mathcal{U}_t(\cdot) : H_0^1 \cap H^2 \to H_0^1 \cap H^2$ be the resolving operator. If $T > 0$, $z_n \in H_0^1 \cap H^2$ and $z_n \rightharpoonup z_0$ in H^2, then $\mathcal{U}_T(z_{n_k}) \rightharpoonup \mathcal{U}_T(z_0)$ in H^2 for some sequence $k_n \geq 1$.*

Sketch of the proof. The local well-posedness of (3.4), (1.2) and (1.3) is standard (see [13]). From the construction of the feedback u it follows that a finite-time blow-up is impossible. Hence the solution is global in time. Let us show that $u(z_n) \to u(z_0)$ for any $z_n \in H_0^1 \cap H^2$ such that $z_n \rightharpoonup z_0$ in H^2. Notice that (3.2) and the fact that Q is real imply that

$$u(z) = -\delta \operatorname{Im}(\langle \alpha Q(-\Delta + V)z, (-\Delta + V)z \rangle) + \tilde{u}(z) = \tilde{u}(z),$$

where

$$\tilde{u}(z) = -\delta \operatorname{Im}(\langle \alpha(-\Delta + V) P_{1,V}(Qz), (-\Delta + V)(-\langle z, e_{1,V} \rangle e_{1,V}) \rangle$$
$$+ \langle \alpha(-\Delta + V)(-\langle Qz, e_{1,V} \rangle e_{1,V}), (-\Delta + V)z \rangle$$
$$+ \langle \alpha(-\nabla Q \cdot \nabla z - z\Delta Q), (-\Delta + V)z \rangle$$
$$- \langle Qz, e_{1,V} \rangle \langle e_{1,V}, z \rangle).$$

Thus $\tilde{u}(z_n) \to \tilde{u}(z_0)$. As $\mathcal{U}_t(z_n)$ is bounded in $C([0, T], H^2)$, for some sequence $k_n \geq 1$ we have $\mathcal{U}.(z_{k_n}) \to U(\cdot)$ in $L^2([0, T], L^2)$ and $\mathcal{U}_t(z_{k_n}) \rightharpoonup U(t)$ in H^2 for all $t \in [0, T]$. Passing to the limit in (3.4), we see that $U(t)$ is a solution of problem (3.4), (1.2), (1.3). Uniqueness gives $U(t) = \mathcal{U}_t(z_0)$. This completes the proof. $\square$

Thus if $z_0, \Delta z_0 \in H_0^1 \cap H^2$, then (3.3) is verified for $z(t) = \mathcal{U}_t(z_0)$. A density argument proves the identity for any $z_0 \in H_0^1 \cap H^2$.

Let us assume that the functions V and Q satisfy the following condition:

Condition 3.2. *The functions* $V, Q \in C^\infty(\overline{D}, \mathbb{R})$ *are such that:*

(i) $\langle Qe_{1,V}, e_{j,V} \rangle \neq 0$ *for all* $j \geq 2$,
(ii) $\lambda_{1,V} - \lambda_{j,V} \neq \lambda_{p,V} - \lambda_{q,V}$ *for all* $j, p, q \geq 1$ *such that* $\{1, j\} \neq \{p, q\}$ *and* $j \neq 1$.

The theorem below is the main result of this section.

Theorem 3.3. *Under Condition* 3.2, *there is a finite or countable set* $J \subset \mathbb{R}_+^*$ *such that for any* $\alpha \notin J$ *and* $z_0 \in S \cap H_0^1 \cap H^2$ *with* $\langle z_0, e_{1,V} \rangle \neq 0$ *and* $0 < \mathcal{V}(z_0) < 1$ *there is a sequence* $k_n \geq 1$ *verifying*

$$\mathcal{U}_{k_n}(z_0) \rightharpoonup ce_{1,V} \quad in\ H^2,$$

where $c \in \mathbb{C}, |c| = 1$.

See SubSect. 3.3 for the proof of this theorem. The following lemma shows that the hypothesis on the initial condition z_0 is not restrictive.

Lemma 3.4. *For any* $z_0 \in S$ *there is a control* $u \in C^\infty([0, \infty), \mathbb{R})$ *and a time* $k \geq 1$ *such that* $\langle \mathcal{U}_k(z_0, u), e_{1,V} \rangle \neq 0$.

Proof. It suffices to find a control u and a time $k \geq 1$ such that

$$\|\mathcal{U}_k(z_0, u) - ce_{1,V}\| < \sqrt{2} \tag{3.5}$$

for some $c \in \mathbb{C}, |c| = 1$. Take any $\hat{z}_0 \in S \cap H_0^1 \cap H^2$ such that $\langle \hat{z}_0, e_{1,V} \rangle \neq 0$ and

$$\|z_0 - \hat{z}_0\| < \frac{\sqrt{2}}{2}.$$

By Theorem 3.3, there is a control $u \in C^\infty([0, \infty), \mathbb{R})$ and a time $k \geq 1$ such that

$$\|\mathcal{U}_k(\hat{z}_0, u) - ce_{1,V}\| < \frac{\sqrt{2}}{2}.$$

Using the fact that the L^2-distance between two solutions of (1.1), (1.2) with the same control is constant, we obtain (3.5). □

3.2. Approximate controllability. Before proving Theorem 3.3, let us give an application of the result. For any $d > 0$ define the set

$$C_d = \{u \in C^\infty([0, \infty), \mathbb{R}) : \sup_{t \in [0,\infty)} |u(t)| < d\}.$$

We say that problem (1.1), (1.2) is approximately controllable in L^2 at integer times if for any $\varepsilon, d > 0$ and for any points $z_0, z_1 \in S$ there is a time $k \in \mathbb{N}$ and a control $u \in C_d$ such that

$$\|\mathcal{U}_k(z_0, u) - z_1\| < \varepsilon.$$

Theorem 3.5. *Under Condition* 3.2, *problem* (1.1), (1.2) *is approximately controllable in* L^2 *at integer times.*

Proof. Theorem 3.3 implies that for any $z \in S \cap H_0^1 \cap H^2$ there is $u \in C_d$ such that

$$\|\mathcal{U}_k(z, u) - e_{1,V}\| < \frac{\varepsilon}{2} \tag{3.6}$$

for some $k \geq 1$. As the L^2-distance between two solutions of (1.1), (1.2) with the same control is constant, by a density argument, we get that for any $z \in S$ a control $u \in C_d$ exists such that (3.6) holds.

Here we need the following result often referred to as time reversibility property of the Schrödinger equation.

Lemma 3.6. *Suppose that* $\mathcal{U}_k(\bar{z}, w) = \bar{y}$ *for some* $z \in L^2$, $w \in C_d$ *and* $k \geq 1$. *Then* $\mathcal{U}_k(y, u) = z$, *where* $u(t) = w(k - t)$.

The proof of this lemma is clear. Let us fix any $z_0, z_1 \in S$ and let $u_0, w \in C_d$ be such that

$$\|\mathcal{U}_{k_1}(\bar{z}_1, w) - e_{1,V}\| < \frac{\varepsilon}{2},$$
$$\|\mathcal{U}_{k_0}(z_0, u_0) - e_{1,V}\| < \frac{\varepsilon}{2}$$

for some $k_0, k_1 \geq 1$. Define $y := \overline{\mathcal{U}_{k_1}(\bar{z}_1, w)}$. Then by Lemma 3.6, we have $\mathcal{U}_{k_1}(y, u_1) = z_1$, where $u_1(t) := w(k_1 - t)$. Again using the fact that the L^2-distance between two solutions of (1.1), (1.2) with the same control is constant, we get

$$\|\mathcal{U}_{k_1}(e_{1,V}, u_1) - z_1\| = \|e_{1,V} - y\| < \frac{\varepsilon}{2}.$$

Taking $k = k_0 + k_1$ and $\hat{u}(t) = u_0(t), t \in [0, k_0)$ and $\hat{u}(t) = u_1(t - k_0), t \in [k_0, \infty)$, we obtain

$$\|\mathcal{U}_k(z_0, \hat{u}) - z_1\| < \varepsilon.$$

Finally, using the continuity of $\mathcal{U}_k(z_0, \cdot)$, we find $u \in C_d$ satisfying

$$\|\mathcal{U}_k(z_0, u) - z_1\| < \varepsilon.$$

$\square$

Remark 3.7. We note that for $m = 1$, $Q(x) = x$ a stronger result is obtained by K. Beauchard and M. Mirrahimi [10] in the case of the space L^2. They prove a result of approximate stabilization of eigenstates. The proof of this result remains literally the same for system (1.1), (1.2) under Condition 3.2. One should just pay attention to the fact that in the case of any space dimension m the spectral gap property for the eigenvalues used in [10] does not hold. The argument can be replaced by Lemma 3.10.

3.3. Proof of Theorem 3.3.

Step 1. Let us suppose that $u(\mathcal{U}_t(z_0)) = 0$ for all $t \geq 0$. Then

$$\mathcal{U}_t(z_0) = \sum_{j=1}^{\infty} e^{-i\lambda_{j,V}t} \langle z_0, e_{j,V} \rangle e_{j,V}. \tag{3.7}$$

Substituting (3.7) into (3.2), we get

$$
\begin{aligned}
0 = {} & \sum_{j=1,k=2}^{\infty} \alpha\lambda_{k,V} \langle z_0, e_{j,V} \rangle \langle e_{k,V}, z_0 \rangle \langle (-\Delta + V)(P_{1,V}(Qe_{j,V})), e_{k,V} \rangle e^{-i(\lambda_{j,V}-\lambda_{k,V})t} \\
& - \sum_{j=1,k=2}^{\infty} \alpha\lambda_{k,V} \langle e_{j,V}, z_0 \rangle \langle z_0, e_{k,V} \rangle \langle e_{k,V}, (-\Delta + V)(P_{1,V}(Qe_{j,V})) \rangle e^{i(\lambda_{j,V}-\lambda_{k,V})t} \\
& - \sum_{j=1}^{\infty} \langle z_0, e_{j,V} \rangle \langle e_{1,V}, z_0 \rangle \langle Qe_{j,V}, e_{1,V} \rangle e^{i(\lambda_{1,V}-\lambda_{j,V})t} \\
& + \sum_{j=1}^{\infty} \langle e_{j,V}, z_0 \rangle \langle z_0, e_{1,V} \rangle \langle Qe_{j,V}, e_{1,V} \rangle e^{-i(\lambda_{1,V}-\lambda_{j,V})t} \\
= {} & \sum_{j=2,k=2}^{\infty} P(z_0, Q, j, k) e^{-i(\lambda_{j,V}-\lambda_{k,V})t} \\
& + \sum_{j=2}^{\infty} \Big[(\alpha\lambda_{j,V} \langle e_{j,V}, (-\Delta + V)(P_{1,V}(Qe_{1,V})) \rangle + \langle Qe_{j,V}, e_{1,V} \rangle) \\
& \times \langle z_0, e_{1,V} \rangle \langle e_{j,V}, z_0 \rangle e^{-i(\lambda_{1,V}-\lambda_{j,V})t} \Big] \\
& - \sum_{j=2}^{\infty} \Big[(\alpha\lambda_{j,V} \langle (-\Delta + V)(P_{1,V}(Qe_{1,V})), e_{j,V} \rangle + \langle Qe_{j,V}, e_{1,V} \rangle) \\
& \times \langle e_{1,V}, z_0 \rangle \langle z_0, e_{j,V} \rangle e^{i(\lambda_{1,V}-\lambda_{j,V})t} \Big], \tag{3.8}
\end{aligned}
$$

where $P(z_0, Q, j, k)$ is a constant. In view of Condition 3.2, (ii), Lemma 3.10 below implies that the coefficients of exponential functions in (3.8) vanish. Condition 3.2, (i), implies that the set

$$
\begin{aligned}
J := \{ \alpha \in \mathbb{R} : {} & \alpha\lambda_{j,V} \langle (-\Delta + V)(P_{1,V}(Qe_{1,V})), e_{j,V} \rangle + \langle Qe_{j,V}, e_{1,V} \rangle = 0 \\
& \text{for some } j \geq 2 \}
\end{aligned}
$$

is finite or countable. Thus we get that $z_0 = ce_{1,V}$ for some $c \in \mathbb{C}$, $|c| = 1$ which is in contradiction to $\mathcal{V}(z_0) > 0$. Thus there is a time $t_0 > 0$ such that $u(\mathcal{U}_{t_0}(z_0)) \neq 0$ and

$$\mathcal{V}(\mathcal{U}_k(z_0)) - \mathcal{V}(z_0) = -\frac{2}{\delta} \int_0^k u^2(\mathcal{U}_s(z_0)) \mathrm{d}s < 0$$

for any $k \geq t_0$.

Step 2. Let $\mathcal{K}$ be the H^2-weak ω-limit set of the trajectory for (3.4), (1.2) issued from z_0, i.e.

$$\mathcal{K} := \{z \in H_0^1 \cap H^2 : \mathcal{U}_{k_n}(z_0) \rightharpoonup z \text{ in } H^2 \text{ for some } k_n \to \infty\}.$$

Let

$$m := \inf_{z \in \mathcal{K}} \mathcal{V}(z).$$

This infimum is attained, i.e. there is $e \in \mathcal{K}$ such that

$$\mathcal{V}(e) = \inf_{z \in \mathcal{K}} \mathcal{V}(z).$$

Indeed, take any minimizing sequence $z_n \in \mathcal{K}$, so that $\mathcal{V}(z_n) \to m$. By (3.1), z_n is bounded in H^2. Thus, without loss of generality, we can assume that $z_n \rightharpoonup e$ in H^2. This implies that $\mathcal{V}(e) \leq \liminf_{n \to \infty} \mathcal{V}(z_n) = m$. Let us show that $e \in \mathcal{K}$. We can choose a sequence $k_n \geq 1$ such that

$$\|\mathcal{U}_{k_n}(z_0) - z_n\| \leq \frac{1}{n}. \tag{3.9}$$

As $\mathcal{U}_{k_n}(z_0)$ is bounded in H^2, without loss of generality, we can suppose that $\mathcal{U}_{k_n}(z_0) \rightharpoonup \tilde{e}$, $\tilde{e} \in S \cap H_0^1 \cap H^2$. Clearly, (3.9) implies that $e = \tilde{e}$, hence $e \in \mathcal{K}$ and $\mathcal{V}(e) = m$.

Let us show that $\mathcal{V}(e) = 0$. Suppose that $\mathcal{V}(e) > 0$. As $\mathcal{V}(e) \leq \mathcal{V}(z_0) < 1$, we have $\langle e, e_{1,V} \rangle \neq 0$. Then, by Step 1, there is a time $k \geq 1$ such that $\mathcal{V}(\mathcal{U}_k(e)) < \mathcal{V}(e)$. Proposition 3.1 implies that $\mathcal{U}_k(e) \in \mathcal{K}$. This contradicts the definition of e. Hence $\mathcal{V}(e) = 0$. Thus $e = ce_{1,V}$, $|c| = 1$ and $ce_{1,V} \in \mathcal{K}$.

Remark 3.8. We note that if there is a sequence $n_k \geq 1$ such that $\mathcal{U}_{n_k}(z_0)$ converges in H^2 and z_0 satisfies the hypotheses of Theorem 3.3, then the proof of the stabilization result obtained in [9] for finite-dimensional approximations of the Schrödinger equation works giving

$$\mathcal{U}_{n_k}(z_0, u) \to e_{1,V} \text{ in } H^2.$$

However, the existence of such a sequence is an open question.

Remark 3.9. Modifying slightly Condition 3.2, Theorem 3.3 can be restated for the eigenfunction $e_{i,V}$, $i \geq 1$. Indeed, one should replace $\lambda_{1,V}$ and $e_{1,V}$ by $\lambda_{i,V}$ and $e_{i,V}$ in Condition 3.2 and use the Lyapunov function

$$\mathcal{V}_i(z) := \alpha \|(-\Delta + V)P_{i,V}z\|^2 + 1 - |\langle z, e_{i,V} \rangle|^2, \quad z \in S \cap H_0^1 \cap H^2, \tag{3.10}$$

where $P_{i,V}$ is the orthogonal projection in L^2 onto the closure of the vector span of $\{e_{k,V}\}_{k \neq i}$.

Lemma 3.10. *Suppose that* $r_j \in \mathbb{R}$ *and* $r_k \neq r_j$ *for* $k \neq j$. *If*

$$\sum_{j=1}^{\infty} c_j e^{ir_j t} = 0 \tag{3.11}$$

for any $t \geq 0$ *and for some sequence* $c_j \in \mathbb{C}$ *such that* $\sum_{j=1}^{\infty} |c_j| < \infty$, *then* $c_j = 0$ *for all* $j \geq 1$.

Proof. Multiplying (3.11) by $e^{-ir_n t}$ and integrating on the interval $[0, T]$, we get

$$c_n = -\frac{1}{T} \sum_{j=1, j \neq n}^{\infty} c_j \int_0^T e^{i(r_j - r_n)t} \mathrm{d}t = -\frac{1}{T} \sum_{j=1, j \neq n}^{\infty} c_j \frac{e^{i(r_j - r_n)T} - 1}{i(r_j - r_n)} \to 0$$

as $T \to \infty$, by the Lebesgue theorem on dominated convergence.   □

3.4. Genericity of Condition 3.2. Let us recall some definitions. Let X be a complete metric space and $A \subset X$. Then A is said to be a G_δ set if it is a countable intersection of dense open sets. It follows from the Baire theorem that any G_δ subset is dense. A set $B \subset X$ is called residual if it contains a G_δ subset.

Let us endow the space $C^\infty(\overline{D}, \mathbb{R})$ with its usual topology given by the countable family of norms:

$$p_n(Q) := \sum_{|\alpha| \leq n} \sup_{x \in D} |\partial^\alpha Q(x)|.$$

The set $\mathcal{P}$ of all functions $Q \in C^\infty(\overline{D}, \mathbb{R})$ such that property (i) of Condition 3.2 is verified is a G_δ. Indeed, let us fix an integer $j \geq 1$ and let $\mathcal{P}_j$ be the set of functions $Q \in C^\infty(\overline{D}, \mathbb{R})$ verifying $\langle Q e_{1,V}, e_{j,V} \rangle \neq 0$. The unique continuation theorem for the operator $-\Delta + V$ (see [19]) implies that there is a ball $B \subset D$ such that $e_{1,V}(x) e_{j,V}(x) \neq 0$ for all $x \in B$. Let $Q \in C^\infty(\overline{D}, \mathbb{R})$ be such that $Q \neq 0$, supp $Q \subset B$ and $Q \geq 0$. Then $Q \in \mathcal{P}_j$, hence $\mathcal{P}_j$ is non-empty. Clearly, $\mathcal{P}_j$ is open. Take any $Q_1 \in C^\infty(\overline{D}, \mathbb{R})$ such that $\langle Q_1 e_{1,V}, e_{j,V} \rangle = 0$ and $Q_2 \in \mathcal{P}_j$. Then $\langle (Q_1 + \tau Q_2) e_{1,V}, e_{j,V} \rangle \neq 0$ for all $\tau \neq 0$. Thus $\mathcal{P}_j$ is dense in $C^\infty(\overline{D}, \mathbb{R})$ and $\mathcal{P} = \cap_{j=1}^{\infty} \mathcal{P}_j$ is a G_δ set.

The following lemma shows that property (ii) of Condition 3.2 is generic in the 1D case.

Lemma 3.11. *Let $I \subset \mathbb{R}$ be an interval and let $\mathcal{Q}$ be the set of all functions $V \in C^\infty(I, \mathbb{R})$, verifying*

$$\lambda_{i,V} - \lambda_{j,V} \neq \lambda_{p,V} - \lambda_{q,V} \tag{3.12}$$

for all $i, j, p, q \geq 1$ such that $\{i, j\} \neq \{p, q\}$ and $i \neq j$. Then $\mathcal{Q}$ is a G_δ set.

Proof. It is well known that the spectrum $\{\lambda_{j,V}\}$ of $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V$ is non-degenerate for any $V \in C^\infty(\overline{D}, \mathbb{R})$, and $e_{j,V}$ and $\lambda_{j,V}$ are real-analytic in V (e.g., see [26]). Let us introduce the set $\mathcal{Q}_n$, $n \geq 1$ of all functions $V \in C^\infty(\overline{D}, \mathbb{R})$ such that (3.12) is satisfied for any $1 \leq i, j, p, q \leq n$. Clearly,

$$\mathcal{Q} = \bigcap_{n=1}^{\infty} \mathcal{Q}_n.$$

It suffices to prove that $\mathcal{Q}_n$ is open and dense in $C^\infty(\overline{D}, \mathbb{R})$. The fact that $\mathcal{Q}_n$ is open follows directly from the continuity of $\lambda_{j,V}$ in V. Let us prove that $\mathcal{Q}_n$ is dense in $C^\infty(\overline{D}, \mathbb{R})$.

Take any $1 \leq i, j, p, q \leq n$ such that $\{i, j\} \neq \{p, q\}$ and $i \neq j$, and let $\mathcal{Q}_{i,j,p,q}$ be the set of functions $V \in C^\infty(\overline{D}, \mathbb{R})$ such that (3.12) is satisfied. Suppose we have proved that for any $V \in C^\infty(\overline{D}, \mathbb{R})$ there is $\sigma \in C^\infty(\overline{D}, \mathbb{R})$ such that

$$\lambda_{i,V+\tau\sigma} - \lambda_{j,V+\tau\sigma} \neq \lambda_{p,V+\tau\sigma} - \lambda_{q,V+\tau\sigma}, \tag{3.13}$$

for any small $\tau > 0$. This implies that $\mathcal{Q}_{i,j,p,q}$ is dense. On the other hand, $\mathcal{Q}_{i,j,p,q}$ is open. Hence $\mathcal{Q}_n$ is dense, as

$$\mathcal{Q}_n = \bigcap_{1 \leq i,j,p,q \leq n} \mathcal{Q}_{i,j,p,q}.$$

To prove (3.13), following [2], let us write

$$\lambda_{j,V+\tau\sigma} = \lambda_{j,V} + \alpha_j \tau + \beta_j(\tau)\tau^2, \tag{3.14}$$

$$e_{j,V+\tau\sigma} = e_{j,V} + v_j \tau + w_j(\tau)\tau^2. \tag{3.15}$$

Differentiating the identity

$$(-\frac{d^2}{dx^2} + V + \tau\sigma - \lambda_{j,V+\tau\sigma})e_{j,V+\tau\sigma} = 0$$

with respect to τ at $\tau = 0$ and using (3.14) and (3.15), we get

$$(-\frac{d^2}{dx^2} + V - \lambda_{j,V})v_j + (\sigma - \alpha_j)e_{j,V} = 0.$$

Taking the scalar product of this identity with $e_{j,V}$, we obtain

$$\langle \sigma, |e_{j,V}|^2 \rangle = \alpha_j. \tag{3.16}$$

Suppose that

$$\lambda_{i,V+\tau_n\sigma} - \lambda_{j,V+\tau_n\sigma} = \lambda_{p,V+\tau_n\sigma} - \lambda_{q,V+\tau_n\sigma}$$

for any $\sigma \in C^\infty(\overline{D}, \mathbb{R})$ and for some sequence $\tau_n \to 0$. Clearly, this implies that

$$\alpha_i - \alpha_j = \alpha_p - \alpha_q.$$

In view of (3.16), this gives

$$|e_{i,V}|^2 - |e_{j,V}|^2 = |e_{p,V}|^2 - |e_{q,V}|^2. \tag{3.17}$$

On the other hand, by Theorem 9 in [26] (see p. 46), the system $\{|e_{n,V}|^2\}$ is independent for any $V \in L^2$. This contradiction proves (3.13) and completes the proof of the lemma. $\square$

We now turn to the multidimensional case. Let us assume that $D = [0, 1]^n$ and introduce the space

$$\mathcal{G} := \{V \in C^\infty(D, \mathbb{R}) : V(x_1, \ldots, x_n) = V_1(x_1) + \cdots + V_n(x_n)$$
$$\text{for some } V_k \in C^\infty([0, 1], \mathbb{R}), k = 1, \ldots, n\}.$$

Endow $\mathcal{G}$ with the metric of $C^\infty(D, \mathbb{R})$. It is not difficult to verify that $\mathcal{G}$ is a closed subspace in $C^\infty(D, \mathbb{R})$.

Lemma 3.12. *The set of all functions $V \in \mathcal{G}$, verifying*

$$\lambda_{i,V} - \lambda_{j,V} \neq \lambda_{p,V} - \lambda_{q,V} \tag{3.18}$$

for all $i, j, p, q \geq 1$ such that $\{i, j\} \neq \{p, q\}$ and $i \neq j$, is a G_δ set.

Proof. Notice that any eigenfunction of $-\Delta + V$, $V \in \mathcal{G}$ has the form

$$e_{l,V}(x_1, \ldots, x_n) = e_{l_1,V_1}(x_1) \cdot \cdots \cdot e_{l_n,V_n}(x_n), \tag{3.19}$$

where $e_{l_k,V_k}(x_k)$ is an eigenfunction of the operator $-\frac{d^2}{dx_k^2} + V_k$. Indeed, any function of the form (3.19) is an eigenfunction, and the set of all functions of this form is a basis in $L^2(D)$.

Let $i, j, p, q \geq 1$ be such that $\{i, j\} \neq \{p, q\}$ and $i \neq j$, and let $e_{i_n,V_n}(x_n)$, $e_{j_n,V_n}(x_n)$, $e_{p_n,V_n}(x_n)$ and $e_{q_n,V_n}(x_n)$ be the eigenfunctions in (3.19). Without loss of generality, we can assume that the functions $(e_{i_n,V_n}(x_n))^2$, $(e_{j_n,V_n}(x_n))^2$, $(e_{p_n,V_n}(x_n))^2$ and $(e_{q_n,V_n}(x_n))^2$ are linearly independent (see Theorem 9 in [26]). Any eigenfunction e_{l_k,V_k} has a finite number of zeros in interval $[0, 1]$. Hence, choosing appropriately the point $x^* \in [0, 1]^{n-1}$, we see that the functions $(e_{i,V}(x^*, x_n))^2$, $(e_{j,V}(x^*, x_n))^2$, $(e_{p,V}(x^*, x_n))^2$ and $(e_{q,V}(x^*, x_n))^2$, $x_n \in [0, 1]$ are linearly independent. This implies that relation (3.17) does not hold. Thus the proof of Lemma 3.11 works implying the genericity. □

4. Applications

4.1. Nonlinear Schrödinger equation. Let us consider the nonlinear Schrödinger equation

$$i\dot{z} = -\Delta z + V(x)z + u(t)Q(x)|z|^2 z, \quad x \in D, \tag{4.1}$$

$$z|_{\partial D} = 0, \tag{4.2}$$

$$z(0, x) = z_0(x), \tag{4.3}$$

where $D \subset \mathbb{R}^3$ is a bounded domain with smooth boundary. Problem (4.1)-(4.3) is locally well-posed.

Lemma 4.1. *For any $z_0 \in H_0^1 \cap H^2$ and for any $u \in L_{loc}^1([0, \infty), \mathbb{R})$ there is a time $T > 0$ such that problem (4.1)-(4.3) has a unique solution $z \in C([0, T], H^2)$. Furthermore, the resolving operator $\mathcal{U}_t(\cdot, u) : H_0^1 \cap H^2 \to H_0^1 \cap H^2$ taking z_0 to $z(t)$ satisfies the relation*

$$\|\mathcal{U}_t(z_0, u)\| = \|z_0\|, \quad t \in [0, T].$$

See [13] for the proof. Define $z(t) = \mathcal{U}_t(z_0, u)$ and let us calculate the derivative

$$\frac{d}{dt}\mathcal{V}_i(z(t)) = 2\alpha \operatorname{Re}(\langle(-\Delta + V)P_{i,V}\dot{z}, P_{i,V}(-\Delta + V)z\rangle) - 2\operatorname{Re}(\langle\dot{z}, e_{i,V}\rangle\langle e_{i,V}, z\rangle)$$

$$= 2\alpha \operatorname{Re}(\langle(-\Delta + V)P_{i,V}(i\Delta z - iVz - iuQ|z|^2 z), (-\Delta + V)P_{i,V}z\rangle)$$

$$-2\operatorname{Re}(\langle i\Delta z - iVz - iuQ|z|^2 z, e_{i,V}\rangle\langle e_{i,V}, z\rangle)$$

$$= 2u\operatorname{Im}(\alpha\langle(-\Delta + V)P_{i,V}(Q|z|^2 z), (-\Delta + V)P_{i,V}z\rangle$$

$$-\langle Q|z|^2 z, e_{i,V}\rangle\langle e_{i,V}, z\rangle),$$

where $\mathcal{V}_i$ is defined by (3.10). Take

$$u(z) := -\operatorname{Im}(\langle \alpha(-\Delta + V)P_{i,V}(Q|z|^2 z), (-\Delta + V)P_{i,V}z\rangle - \langle Q|z|^2 z, e_{i,V}\rangle\langle e_{i,V}, z\rangle).$$
(4.4)

Problem (4.1)-(4.3) with feedback (4.4) is globally well-posed in H^2 (cf. Theorem 3.1). Let $\mathcal{U}_t(\cdot) : H_0^1 \cap H^2 \to H_0^1 \cap H^2$ be the resolving operator. In order to formulate the main result, we introduce the following hypothesis:

Condition 4.2. *The functions* V, $Q \in C^\infty(\overline{D}, \mathbb{R})$ *are such that:*

(i) $\langle Qe_{i,V}e_{j,V}, e_{p,V}e_{q,V}\rangle \neq 0$ *for all* $i, j, p, q \geq 1$,
(ii) $\lambda_{i,V} - \lambda_{j,V} + \lambda_{p,V} - \lambda_{q,V} \neq \lambda_{i',V} - \lambda_{j',V} + \lambda_{p',V} - \lambda_{q',V}$ *for all integers* $i, j, p, q, i', j', p', q'$ *such that* $\{i, j, p, q\} \neq \{i', j', p', q'\}$ *and* $\{i, p\} \neq \{j, q\}$.

The theorem below is the version of Theorem 3.3 for system (4.1)-(4.3).

Theorem 4.3. *Under Condition 4.2, there is a finite or countable set* $J \subset \mathbb{R}_+^*$ *such that for any* $\alpha \notin J$, $\mathfrak{l} \geq 1$ *and* $z_0 \in S \cap H_0^1 \cap H^2$ *with* $\langle z_0, e_{l,V}\rangle \neq 0$ *and* $0 < \mathcal{V}_l(z_0) < 1$ *there is a sequence* $k_n \geq 1$ *verifying*

$$\mathcal{U}_{k_n}(z_0) \rightharpoonup ce_{l,V} \quad \text{in } H^2,$$

where $c \in \mathbb{C}$, $|c| = 1$.

The proof of this theorem is very close to that of Theorem 3.3. One should notice that, under Condition 4.2, there is a time $t_0 > 0$ such that $u(\mathcal{U}_{t_0}(z_0, 0)) \neq 0$, and then conclude as in Step 2 in the proof of Theorem 3.3.

Remark 4.4. Notice that, as Eq. (4.1) is nonlinear, the distance between two solutions with the same control is not constant. Hence the proof of approximate controllability given in Theorem 3.5 does not work here.

Lemma 4.5. *For any* $l \geq 1$, $d > 0$ *and* $z_0 \in S$ *there is a control* $u \in C_d$ *and a time* $k \geq 1$ *such that* $\langle \mathcal{U}_k(z_0, u), e_{l,V}\rangle \neq 0$.

Proof. Assume that $\langle z_0, e_{l,V}\rangle = 0$. Let us show that there is a control $u \in C_d$ such that $\langle \mathcal{U}_k(z_0, u), e_{l,V}\rangle \neq 0$ for some $k \geq 0$. As (4.1) is nonlinear, the proof given in Lemma 3.4 does not work.

If $z_0 \notin \{ce_{j,V} : c \in \mathbb{C}, |c| = 1, j \geq 1\}$, then, by Theorem 4.3, there is an integer $p \geq 1$, a sequence $k_n \geq 1$ and a constant $c \in \mathbb{C}$, $|c| = 1$ such that $\mathcal{U}_{k_n}(z_0) \rightharpoonup ce_{p,V}$ in H^2. Hence, without loss of generality, we can assume that $z_0 = e_{p,V}$ for some $p \neq l$. Let us introduce the following two-dimensional subspace of $L^2([0, 1], \mathbb{R})$:

$$E = \{a \sin(\lambda_{p,V} - \lambda_{l,V})t + b \cos(\lambda_{p,V} - \lambda_{l,V})t : a, b \in \mathbb{R}\}.$$

For any $u \in E$, define the mapping $\Phi(u) = \langle \mathcal{U}_1(e_{p,V}, u), e_{l,V}\rangle$, whenever the solution $\mathcal{U}_t(e_{p,V}, u)$ exists up to time $t = 1$. Notice that $\Phi(0) = \langle e^{-i\lambda_{p,V}}e_{p,V}, e_{l,V}\rangle = 0$, hence Φ is well defined in a neighborhood of $0 \in E$. We are going to show that the conditions of the inverse mapping theorem are satisfied in a neighborhood of the point $0 \in E$. Clearly, Φ is continuously differentiable. Let us show that the mapping $D\Phi(0)$:

$E \to \mathbb{C}$ is an isomorphism. Consider the linearization of (4.1), (4.2), $z_0 = e_{p,V}$ around $(e^{-i\lambda_{p,V}t}e_{p,V}, 0)$:

$$i\dot{y} = -\Delta y + V(x)y + u(t)Q(x)e_{p,V}^3 e^{-i\lambda_{p,V}t}, \quad x \in D, \tag{4.5}$$

$$y|_{\partial D} = 0, \tag{4.6}$$

$$y(0) = 0. \tag{4.7}$$

One can verify that $D\Phi(0)(u) = \langle y(1), e_{l,V} \rangle$. System (4.5)-(4.7) is equivalent to

$$y = -i \int_0^t e^{-i\lambda_{p,V}s} u(s) S(t-s)(Qe_{p,V}^3) ds, \tag{4.8}$$

where $S(t)$ is the unitary group associated with $i\Delta - iV$. Taking the scalar product of (4.8) with $e_{l,V}$, we obtain for $t = 1$,

$$\langle y, e_{l,V} \rangle = -ie^{-i\lambda_{l,V}} \langle Qe_{p,V}^3, e_{l,V} \rangle \int_0^1 e^{-i(\lambda_{p,V}-\lambda_{l,V})s} u(s) ds.$$

Condition 4.2 implies that $\lambda_{p,V} - \lambda_{l,V} \neq 0$, hence $D\Phi(0) : E \to \mathbb{C}$ is an isomorphism. Applying the inverse mapping theorem, we conclude that Φ is a C^1 diffeomorphism in a neighborhood of $0 \in E$. Thus there is a control $u \in C_d$ such that $\langle \mathcal{U}_1(e_{p,V}, u), e_{l,V} \rangle \neq 0$. $\square$

4.2. Randomly forced Schrödinger equation.

4.2.1. Growth of Sobolev norms Let us consider the problem

$$i\dot{z} = -\Delta z + V(x)z + \beta(t)Q(x)z, \quad x \in D, \tag{4.9}$$

$$z|_{\partial D} = 0, \tag{4.10}$$

$$z(0) = z_0, \tag{4.11}$$

where $V, Q \in C^\infty(\overline{D}, \mathbb{R})$ are given functions. We assume that $\beta(t)$ is a random process of the form (2.2), where the random variables η_k verify the following condition:

Condition 4.6. *The random variables η_k have the form*

$$\eta_k(t) = \sum_{j=1}^{\infty} b_j \xi_{jk} g_j(t), \quad t \in [0, 1],$$

where $\{g_j\}$ is an orthonormal basis in $L^2([0, 1], \mathbb{R})$, $b_j > 0$ are constants with

$$\sum_{j=1}^{\infty} b_j^2 < \infty,$$

and ξ_{jk} are independent real-valued random variables such that $\mathbb{E}\xi_{jk}^2 = 1$. Moreover, the distribution of ξ_{jk} possesses a continuous density ρ_j with respect to the Lebesgue measure and $\rho_j(r) > 0$ for all $r \in \mathbb{R}$.

Notice that this condition in particular implies that

$$\mathbb{P}\{\|u - \beta\|_{L^2([0,l])} < \varepsilon\} > 0$$

for any $u \in L^2([0,l])$ and $\varepsilon > 0$. Moreover, using the continuity of the mapping $\mathcal{U}_l(z_0, \cdot) : L^2([0,l]) \to L^2(D)$, for any $\delta > 0$ we can find a constant $\varepsilon > 0$ such that

$$\mathbb{P}\{\|\mathcal{U}_l(z_0, \beta) - \mathcal{U}_l(z_0, u)\| < \delta\} \geq \mathbb{P}\{\|u - \beta\|_{L^2([0,l])} < \varepsilon\} > 0.$$

Hence, any point $\mathcal{U}_l(z_0, u)$, $u \in L^2([0,l])$ is in the support of the measure $\mathcal{D}(\mathcal{U}_l(z_0, \beta))$.

The following theorem is the main result of this section.

Theorem 4.7. *Suppose that Conditions* 3.2 *and* 4.6 *are satisfied. Then for any* $s > 0$ *and* $z \in H^s \setminus \{0\}$ *we have*

$$\mathbb{P}\{\limsup_{k \to \infty} \|\mathcal{U}_k(z, \beta)\|_s = \infty\} = 1. \tag{4.12}$$

4.2.2. Proof of Theorem 4.7. By Theorem 3.5, system (1.1), (1.2) is approximately controllable at integer times. Since the equation is linear in z, it suffices to prove (4.12) for any $z \in S \cap H^s$. Without loss of generality, we can assume that $s \in (0, 2]$.

Step 1. Let us fix a constant $r > 0$ and introduce the stopping time

$$\tau_r(z) = \min\{k \geq 0 : \mathcal{U}_k(z, \beta) \in B_{H^{-s}}(0, r)\}, \quad z \in B_{L^2}(0, 1).$$

Then we have

$$\mathbb{P}\{\tau_r(z) < \infty\} = 1. \tag{4.13}$$

Indeed, choose an arbitrary point $z' \in S \cap B_{H^{-s}}(0, r)$. By the property of approximate controllability in L^2, there is a control $u \in C_d$ such that $\mathcal{U}_l(z, u)$ is sufficiently close to z' in L^2, hence $\mathcal{U}_l(z, u) \in B_{H^{-s}}(0, r)$. As $\mathcal{U}_l(z, u)$ is in the support of measure $\mathcal{D}(\mathcal{U}_l(z, \beta))$, we have

$$\mathbb{P}\{\mathcal{U}_l(z, \beta) \in B_{H^{-s}}(0, r)\} > 0.$$

Using the continuity of the resolving operator in negative Sobolev norms, we see that there is an H^{-s}-neighborhood $\mathcal{O} = \mathcal{O}(z)$ of z such that

$$\sup_{y \in \mathcal{O}} \mathbb{P}\{\tau_r(y) > l\} < 1.$$

From the compactness of $B_{L^2}(0, 1)$ in H^{-s} it follows that there is a time $k \geq 1$ such that

$$a := \sup_{y \in B_{L^2}(0,1)} \mathbb{P}\{\tau_r(y) > k\} < 1. \tag{4.14}$$

Using the Markov property and (4.14), we obtain

$$\mathbb{P}\{\tau_r(y) > nk\} = \mathbb{E}(I_{\{\tau_r(y) > (n-1)k\}} \mathbb{P}\{\tau_r(x) > k\}|_{x = \mathcal{U}_{(n-1)k}(y, \beta)})$$
$$\leq a\mathbb{P}\{\tau_r(y) > (n-1)k\}. \tag{4.15}$$

Hence

$$\mathbb{P}\{\tau_r(y) > nk\} \leq a^n.$$

Using the Borel–Cantelli lemma, we arrive at (4.13).

Step 2. Take any $z \in S \cap H^s$. Choosing $r = \frac{1}{n}$ and using (4.13), we get

$$\mathbb{P}\{\liminf_{k \to \infty} \|\mathcal{U}_k(z, \beta)\|_{-s} = 0\} = 1. \tag{4.16}$$

Define the event

$$\mathcal{A} := \{\omega \in \Omega : \limsup_{k \to \infty} \|\mathcal{U}_k(z, \beta)\|_s < \infty\}.$$

Suppose that

$$\mathbb{P}\{\mathcal{A}\} > 0.$$

By (4.16), for almost any $\omega \in \mathcal{A}$ there is a sequence $n_k \to \infty$ such that

$$\lim_{n \to \infty} \|\mathcal{U}_{n_k}(z, \beta)\|_{-s} = 0. \tag{4.17}$$

On the other hand, for any $\omega \in \mathcal{A}$, there is a subsequence of n_k (which is also denoted by n_k) and an element $w \in S$ such that

$$\|\mathcal{U}_{n_k}(z, \beta) - w\| \to 0.$$

This contradicts (4.17). Thus $\mathbb{P}\{\mathcal{A}\} = 0$.

Remark 4.8. In view of Theorem 4.3, under Condition 4.2, Theorem 4.7 also holds in the case of nonlinear Equation (4.1). The proof is literally the same. One should just pay attention to the fact that, as in this case finite time blow-up is possible, the restriction of the solution at integer times forms a Markov chain with values in $H^s \cup \{\infty\}$ (e.g., see [28]).

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