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Global approximate controllability for Schrödinger equation in higher Sobolev norms and applications

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Abstract

We prove that the Schrödinger equation is approximately controllable in Sobolev spaces H^s, $s>0$, generically with respect to the potential. We give two applications of this result. First, in the case of one space dimension, combining our result with a local exact controllability property, we get the global exact controllability of the system in higher Sobolev spaces. Then we prove that the Schrödinger equation with a potential which has a random time-dependent amplitude admits at most one stationary measure on the unit sphere S in L^2.

1. Introduction

In this paper, we study the problem

$$i\dot{z} = -\Delta z + V(x)z + u(t)Q(x)z, \quad x \in D, \tag{1.1}$$

$$z|_{\partial D} = 0, \tag{1.2}$$

$$z(0,x) = z_0(x), \tag{1.3}$$

where $D \subset \mathbb{R}^m$ is a bounded domain with smooth boundary, $V \in C^\infty(\overline{D}, \mathbb{R})$ is an arbitrary given function, u is the control, and z is the state. We prove that this system is approximately controllable to the eigenfunctions of $-\Delta + V$ in Sobolev spaces H^s, $s>0$ generically with respect to the dipolar moment Q. In the case $m=1$ and $V=0$, combination of our result with the local exact controllability property obtained by Beauchard [7,8] gives the global exact controllability of the system in the spaces $H^{5+\varepsilon}$. Approximate controllability property implies also that the random Schrödinger equation admits at most one stationary measure on the unit sphere S in L^2.

The problem of controllability of the Schrödinger equation has been largely studied in the literature. Let us mention some previous results closely related to the present paper. Ball, Marsden and Slemrod [5] and Turinici [35] show that the set of attainable points from any initial data in $S \cap H^2$ by system (1.1), (1.2) admits a dense complement in $S \cap H^2$. In [7], Beauchard proves an exact controllability result for the system with $m=1$, $D=(-1,1)$, $V(x)=0$ and $Q(x)=x$ in H^7-neighborhoods of the eigenstates. Beauchard and Coron [9] established later a partial global exact

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controllability result, showing that the system in question is also controlled between some neighborhoods of any two eigenstates. Chambrion et al. [14] and Privat and Sigalotti [31] prove that (1.1), (1.2) is approximately controllable in L^2 generically with respect to the functions V, Q and the domain D. See also the papers [32,36,4,3,1,10] for controllability of finite-dimensional systems and the papers [24,25,6,38,16,26,11,17] for controllability properties of various Schrödinger systems.

Let us recall that, in the case of the space H^2, we established a stabilization property for system (1.1), (1.2) in [28]. Namely, we introduce a Lyapunov function $\mathcal{V}(z) \geqslant 0$ that controls the H^2-norm of z and possesses the following properties:

- $\mathcal{V}(z) = 0$ if and only if $z = ce_{1,V}$, where $e_{1,V}$ is the first eigenfunction of the operator $-\Delta + V$,
- $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}(z(t)) = uG(z(t))$, where $z(t) = z(t,u)$ is the solution of (1.1)–(1.3) and $G(z)$ is a function given explicitly.

Choosing the feedback low $u(z) = -G(z)$, we see that $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}(z(t)) = -u^2 \leqslant 0$, i.e., the function $\mathcal{V}$ decreases on the trajectories of (1.1) corresponding to the feedback u. Thus, to conclude, it suffices to prove that $\mathcal{V}(z(t)) \to 0$. To this end, we use an iteration argument and show that the H^2-weak ω-limit set of any trajectory $z(t)$ contains a minimum point of the function $\mathcal{V}$, i.e., the eigenfunction $ce_{1,V}$, where $c \in \mathbb{C}$, $|c| = 1$. Thus we construct explicitly a feedback low $u(z)$ which forces the trajectories of the system to converge to the eigenstate $\{ce_{1,V}\colon c \in \mathbb{C}, |c| = 1\}$.

The aim of this paper is to generalize these ideas to the case of the spaces H^k, $k > 2$. The main difficulty is that we are not able to construct a Lyapunov function $\mathcal{V}(z)$ such that $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}(z(t)) = uG(z(t))$ for some function G. However, notice that for any $w \in C^\infty([0,T],\mathbb{R})$ we can calculate explicitly the derivative $\frac{\mathrm{d}}{\mathrm{d}\sigma}\mathcal{V}(z(t,\sigma w))|_{\sigma=0}$. We show that there is a time $T > 0$ and a control w such that $\frac{\mathrm{d}}{\mathrm{d}\sigma}\mathcal{V}(z(t,\sigma w))|_{\sigma=0} \neq 0$. Hence we can choose σ_0 close to zero such that

$$\mathcal{V}\big(z(T,\sigma_0 w)\big) < \mathcal{V}\big(z(T,0)\big) = \mathcal{V}(z_0).$$

Thus for any point z_0 we find a time $T > 0$ and a control u such that

$$\mathcal{V}\big(z(T,u)\big) < \mathcal{V}(z_0).$$

Using an iteration argument close to that of [28], we conclude that there are sequences $T_n > 0$ and $u_n \in C^\infty([0,T_n],\mathbb{R})$ such that $z(t_n,u_n) \to e_{1,V}$. Thus any point z_0 can be approximately controlled to $e_{1,V}$.

Then, in the case $m = 1$ and $V = 0$, combining this controllability property with the local exact controllability result obtained by Beauchard [8], we see that the system is globally exactly controllable in $S \cap H^{5+\varepsilon}$ generically with respect to $Q \in C^\infty(\overline{D},\mathbb{R})$, i.e., for any $z_0, z_1 \in S \cap H^{5+\varepsilon}$ there is a time $T > 0$ and a control $u \in H_0^1([0,T],\mathbb{R})$ such that $z(0,u) = z_0$ and $z(T,u) = z_1$.

Next we apply approximate controllability property to prove that the random Schrödinger equation has at most one stationary measure on S. This follows from uniform Feller property and irreducibility of the transition functions of the Markov chain associated to the system in question. Existence of a stationary probability different from the Dirac measure concentrated at zero is an open problem. There are several results on the existence of stationary measures for deterministic Schrödinger equations. Bourgain [12] and Tzvetkov [37] prove the existence of stationary Gibbs measures for different nonlinear Schrödinger systems. Kuksin and Shirikyan [23] construct a stationary measure as a limit of the unique stationary measure of the randomly forced complex Ginzburg–Landau equation when the viscosity goes to zero. In [15], Debussche and Odasso prove existence and uniqueness of stationary measure and a polynomial mixing property for a damped 1D Schrödinger equation. For finite-dimensional approximations of the Schrödinger equation, existence and uniqueness of stationary measure and an exponential mixing property is obtained in [27].

1.1. Notation

In this paper we use the following notation. Let $D \subset \mathbb{R}^m$, $m \geqslant 1$, be a bounded domain with smooth boundary. Let $H^s := H^s(D)$ be the Sobolev space of order $s \geqslant 0$ endowed with the norm $\|\cdot\|_s$. Consider the operators $-\Delta z + Vz$, $z \in \mathcal{D}(-\Delta + V) := H_0^1 \cap H^2$, where $V \in C^\infty(\overline{D},\mathbb{R})$. We denote by $\{\lambda_{j,V}\}$ and $\{e_{j,V}\}$ the sets of eigenvalues and normalized eigenfunctions of $-\Delta + V$. Define the spaces $H^s_{(V)} := D((-\Delta + V)^{\frac{s}{2}})$. Let $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ be the scalar product and the norm in the space L^2. Let S be the unit sphere in L^2. For a Polish space X, we shall use the following notation.

$\mathcal{B}(X)$ is the σ-algebra of Borel subsets of X.
$C(X)$ is the space of real-valued continuous functions on X.
$C_b(X)$ is the space of bounded functions $f \in C(X)$.
$\mathcal{L}(X)$ is the space of functions $f \in C_b(X)$ such that

$$\|f\|_{\mathcal{L}} := \|f\|_\infty + \sup_{u \neq v} \frac{|f(u) - f(v)|}{\|u - v\|} < +\infty.$$

$\mathcal{P}(X)$ is the set of probability measures on $(X, \mathcal{B}(X))$.

2. Main results

2.1. Approximate controllability to $e_{1,V}$

The following lemma shows the well-posedness of system (1.1), (1.2) in Sobolev spaces H^s, $s \geqslant 0$.

Lemma 2.1. *For any control $u \in C^\infty([0,\infty), \mathbb{R})$ such that $\frac{\mathrm{d}^k u}{\mathrm{d}t^k}(0) = 0$ for all $k \geqslant 1$ and for any $z_0 \in H^s_{(V+u(0)Q)}$ problem* (1.1)–(1.3) *has a unique solution $z \in C([0,\infty), H^s)$. Furthermore, if $\frac{\mathrm{d}^k u}{\mathrm{d}t^k}(T) = 0$ for all $k \geqslant 1$, then $z(T) \in H^s_{(V+u(T)Q)}$.*

See [7] for the proof. Denote by $\mathcal{U}_t(\cdot, u) : L^2 \to L^2$ the resolving operator of (1.1), (1.2). As in [28], we assume that the functions V and Q satisfy the following condition.

Condition 2.2. The functions $V, Q \in C^\infty(\overline{D}, \mathbb{R})$ are such that:

(i) $\langle Q e_{1,V}, e_{j,V} \rangle \neq 0$ for all $j \geqslant 2$,
(ii) $\lambda_{1,V} - \lambda_{j,V} \neq \lambda_{p,V} - \lambda_{q,V}$ for all $j, p, q \geqslant 1$ such that $\{1, j\} \neq \{p, q\}$ and $j \neq 1$.

We say that problem (1.1), (1.2) is approximately controllable to $e_{1,V}$ in $H^s, s > 0$ if for any integer $k \geqslant 1$, any constants $\varepsilon, \delta, d > 0$ and for any point $z_0 \in S \cap H^s_{(V)}$ there is a time $T > 0$ and a control $u \in C_0^\infty((0,T), \mathbb{R})$, $\|u\|_{C^k([0,T])} < d$ such that

$$\left\|\mathcal{U}_T(z_0, u) - e_{1,V}\right\|_{s-\delta} < \varepsilon.$$

The theorem below is one of the main results of the present paper.

Theorem 2.3. *Under Condition* 2.2, *for any $s > 0$, problem* (1.1), (1.2) *is approximately controllable to $e_{1,V}$ in H^s.*

In many relevant examples, the spectrum of the operator $-\Delta + V$ is degenerate, hence property (ii) in Condition 2.2 is not verified. In fact, the proof of Theorem 2.3 can be adapted to show the approximate controllability of (1.1), (1.2) for any potential V, under stronger assumptions on the function Q. More precisely, we have the following result.

Theorem 2.4. *Let $V \in C^\infty(\overline{D}, \mathbb{R})$ be arbitrary. The set of functions Q such that problem* (1.1), (1.2) *is approximately controllable to $e_{1,V}$ in H^s for any $s > 0$ is residual in $C^\infty(\overline{D}, \mathbb{R})$, i.e., contains a countable intersection of open dense sets in $C^\infty(\overline{D}, \mathbb{R})$.*

Here $C^\infty(\overline{D}, \mathbb{R})$ is endowed with its usual topology given by the countable family of norms:

$$p_n(Q) := \sum_{|\alpha| \leqslant n} \sup_{x \in D} \left|\partial^\alpha Q(x)\right|.$$

See Section 2.3 for the proof this theorem.

We say that problem (1.1), (1.2) is approximately controllable in L^2 if for any integer $k \geqslant 1$, any constants $\varepsilon, d > 0$ and for any points $z_0, z_1 \in S$ there is a time $T > 0$ and a control $u \in C_0^\infty((0,T), \mathbb{R})$, $\|u\|_{C^k([0,T])} < d$ such that

$$\left\|\mathcal{U}_k(z_0, u) - z_1\right\| < \varepsilon.$$

Combination of Theorem 2.4 with the time reversibility property of the Schrödinger equation implies approximate controllability in L^2.

Theorem 2.5. *Let $V \in C^\infty(\overline{D}, \mathbb{R})$ be arbitrary. The set of functions Q such that problem* (1.1), (1.2) *is approximately controllable in L^2 is residual in $C^\infty(\overline{D}, \mathbb{R})$.*

This result is proved exactly in the same way as Theorem 3.5 in [28].

Proof of Theorem 2.3. By Lemma 3.4 in [28], it suffices to prove the theorem for any initial data $z_0 \in S \cap H^s_{(V)}$ with $\langle z_0, e_{1,V}\rangle \neq 0$. Let us introduce the Lyapunov function

$$\mathcal{V}(z) := \alpha \left\| (-\Delta + V)^{\frac{s}{2}} P_{1,V} z \right\|^2 + 1 - \left|\langle z, e_{1,V}\rangle\right|^2, \quad z \in S \cap H^s_{(V)}, \tag{2.1}$$

where $\alpha > 0$ and $P_{1,V} z := z - \langle z, e_{1,V}\rangle e_{1,V}$ is the orthogonal projection in L^2 onto the closure of the vector span of $\{e_{k,V}\}_{k\geqslant 2}$. Notice that $\mathcal{V}(z) \geqslant 0$ for all $z \in S \cap H^s_{(V)}$ and $\mathcal{V}(z) = 0$ if and only if $z = ce_{1,V}$, $|c| = 1$. For any $z \in S \cap H^s_{(V)}$ we have

$$\mathcal{V}(z) \geqslant \alpha \left\| (-\Delta + V)^{\frac{s}{2}} P_{1,V} z \right\|^2 \geqslant C_1 \|z\|_s^2 - C_2.$$

Thus

$$C\big(1 + \mathcal{V}(z)\big) \geqslant \|z\|_s \tag{2.2}$$

for some constant $C > 0$. We need the following result proved in Section 2.2.

Proposition 2.6. *Fix any constants $s > 0$ and $d > 0$ and an integer $k \geqslant 1$. There is a finite or countable set $J \subset \mathbb{R}^*_+$ such that for any $\alpha \notin J$ and for any $z_0 \in S \cap H^s_{(V)}$ with $\langle z_0, e_{1,V}\rangle \neq 0$ and $0 < \mathcal{V}(z_0)$ there is a time $T > 0$ and a control $u \in C_0^\infty((0,T), \mathbb{R})$, $\|u\|_{C^k([0,T])} < d$ verifying*

$$\mathcal{V}\big(\mathcal{U}_T(z_0, u)\big) < \mathcal{V}(z_0).$$

Let us take any $z_0 \in S \cap H^s_{(V)}$ with $\langle z_0, e_{1,V}\rangle \neq 0$ and $0 < \mathcal{V}(z_0)$, and choose $\alpha > 0$ in (2.1) such that $\mathcal{V}(z_0) < 1$. Define the set

$$\begin{aligned}\mathcal{K} = \big\{ & z \in H^s_{(V)} : \mathcal{U}_{T_n}(z_0, u_n) \to z \text{ in } H^{s-\delta} \text{ for some } T_n \geqslant 0, \\ & u_n \in C_0^\infty\big((0, T_n), \mathbb{R}\big), \|u_n\|_{C^k([0,T_n])} < d \text{ and for any } \delta > 0\big\}.\end{aligned}$$

Let

$$m := \inf_{z \in \mathcal{K}} \mathcal{V}(z).$$

This infimum is attained, i.e., there is $e \in \mathcal{K}$ such that

$$\mathcal{V}(e) = \inf_{z \in \mathcal{K}} \mathcal{V}(z). \tag{2.3}$$

Indeed, take any minimizing sequence $z_n \in \mathcal{K}$, $\mathcal{V}(z_n) \to m$. By (2.2), z_n is bounded in H^s. Thus, without loss of generality, we can assume that $z_n \rightharpoonup e$ in H^s for some $e \in H^s_{(V)}$. This implies that $\mathcal{V}(e) \leqslant \liminf_{n\to\infty} \mathcal{V}(z_n) = m$. Let us show that $e \in \mathcal{K}$. As $z_n \in \mathcal{K}$, there are sequences $T_n > 0$ and $u_n \in C_0^\infty((0, T_n), \mathbb{R})$ such that

$$\big\| \mathcal{U}_{T_n}(z_0, u_n) - z_n \big\|_{s-\delta} \leqslant \frac{1}{n}. \tag{2.4}$$

On the other hand, $z_n \to e$ in $H^{s-\delta}$, and (2.4) implies that $\mathcal{U}_{T_n}(z_0, u_n) \to e$ in $H^{s-\delta}$. Thus $e \in \mathcal{K}$ and $\mathcal{V}(e) = m$.

Let us show that $\mathcal{V}(e) = 0$. Suppose, by contradiction, that $\mathcal{V}(e) > 0$. It follows from (2.3) and from the choice of α that $\mathcal{V}(e) \leqslant \mathcal{V}(z_0) < 1$. This shows that $\langle e, e_{1,V}\rangle \neq 0$. Proposition 2.6 implies that there is a time $T > 0$ and a control $u \in C_0^\infty((0,T), \mathbb{R})$ such that

$$\mathcal{V}\big(\mathcal{U}_T(e, u)\big) < \mathcal{V}(e). \tag{2.5}$$

Define $\tilde{u}_n(t) = u_n(t)$, $t \in [0, T_n]$ and $\tilde{u}_n(t) = u(t - T_n)$, $t \in [T_n, T_n + T]$. Then $\tilde{u}_n \in C_0^\infty((0, T_n + T), \mathbb{R})$ and, by continuity in $H^{s-\delta}$ of the resolving operator for (1.1), (1.2) (e.g., see [13]),

$$\mathcal{U}_{T_n+T}(z_0, \tilde{u}_n) \to \mathcal{U}_T(e, u) \quad \text{in } H^{s-\delta}.$$

This implies that $\mathcal{U}_T(e, u) \in \mathcal{K}$. Clearly, (2.5) contradicts (2.3). It follows that $\mathcal{V}(e) = 0$, hence $e = ce_{1,V}$ for some $c := c(e) \in \mathbb{C}$, $|c| = 1$. As $\mathcal{U}_\tau(ce_{1,V}, 0) = e^{i\tau} ce_{1,V} = e_{1,V}$ for $\tau = -\arg(c)$, we see that $e_{1,V} \in \mathcal{K}$. □

2.2. *Proof of Proposition 2.6*

Take any $z_0 \in S \cap H^s_{(V)}$ such that $\langle z_0, e_{1,V}\rangle \neq 0$ and $0 < \mathcal{V}(z_0)$. This implies that also $\langle z_0, e_{p,V}\rangle \neq 0$ for some $p \geqslant 2$. Let us consider the mapping

$$\mathcal{V}\big(\mathcal{U}_T\big(z_0, (\cdot)w\big)\big) : \mathbb{R} \to \mathbb{R},$$
$$\sigma \to \mathcal{V}\big(\mathcal{U}_T(z_0, \sigma w)\big),$$

where $T > 0$ and $w \in C_0^\infty((0, T), \mathbb{R})$. We are going to show that, for an appropriate choice of T and w, we have $\frac{\mathrm{d}\mathcal{V}(\mathcal{U}_T(z_0,\sigma w))}{\mathrm{d}\sigma}|_{\sigma=0} \neq 0$. Clearly,

$$\frac{\mathrm{d}\mathcal{V}(\mathcal{U}_T(z_0, \sigma w))}{\mathrm{d}\sigma}\bigg|_{\sigma=0} = 2\alpha \operatorname{Re}\big\langle(-\Delta + V)^{\frac{s}{2}} P_{1,V}\mathcal{U}_T(z_0, 0), (-\Delta + V)^{\frac{s}{2}} P_{1,V}\mathcal{R}_T(w)\big\rangle - 2\operatorname{Re}\big\langle\mathcal{U}_T(z_0, 0), e_{1,V}\big\rangle\big\langle e_{1,V}, \mathcal{R}_T(w)\big\rangle, \tag{2.6}$$

where $\mathcal{R}_t(\cdot)$ is the resolving operator of problem

$$i\dot{z} = -\Delta z + V(x)z + w(t)Q(x)\mathcal{U}_t(z_0, 0), \quad x \in D, \tag{2.7}$$

$$z|_{\partial D} = 0, \tag{2.8}$$

$$z(0) = 0. \tag{2.9}$$

System (2.7)–(2.9) is the linearization of (1.1), (1.2) around the solution $\mathcal{U}_t(z_0, 0)$. Note that (2.7)–(2.9) is equivalent to

$$z(t) = -i \int_0^t e^{-i(-\Delta+V)(t-\tau)} w(\tau) Q(x)\mathcal{U}_\tau(z_0, 0)\, \mathrm{d}\tau. \tag{2.10}$$

Taking into account the fact that

$$\mathcal{U}_t(z_0, 0) = \sum_{k=1}^\infty e^{-i\lambda_{k,V}t} \langle z_0, e_{k,V}\rangle e_{k,V}, \tag{2.11}$$

we get from (2.10)

$$\big\langle\mathcal{R}_T(w), e_{j,V}\big\rangle = -ie^{-i\lambda_{j,V}T} \sum_{k=1}^\infty \langle z_0, e_{k,V}\rangle\langle Qe_{k,V}, e_{j,V}\rangle \int_0^T e^{-i(\lambda_{k,V}-\lambda_{j,V})\tau} w(\tau)\, \mathrm{d}\tau. \tag{2.12}$$

Replacing (2.11) and (2.12) into (2.6), we get

$$\begin{aligned}\frac{\mathrm{d}\mathcal{V}(\mathcal{U}_T(z_0, \sigma w))}{\mathrm{d}\sigma}\bigg|_{\sigma=0} &= -2\alpha \operatorname{Im} \sum_{j=2,k=1}^\infty \lambda^s_{j,V}\langle z_0, e_{j,V}\rangle\langle e_{k,V}, z_0\rangle\langle Qe_{k,V}, e_{j,V}\rangle \times \int_0^T e^{i(\lambda_{k,V}-\lambda_{j,V})\tau} w(\tau)\, \mathrm{d}\tau \\ &\quad + 2\operatorname{Im}\sum_{k=1}^\infty \langle z_0, e_{1,V}\rangle\langle e_{k,V}, z_0\rangle\langle Qe_{k,V}, e_{1,V}\rangle \int_0^T e^{i(\lambda_{k,V}-\lambda_{1,V})\tau} w(\tau)\, \mathrm{d}\tau \\ &= \int_0^T \Phi(\tau)w(\tau)\, \mathrm{d}\tau,\end{aligned}$$

where

$$
\begin{aligned}
i\Phi(\tau) :=& -\alpha \sum_{j=2,k=1}^{\infty} \lambda_{j,V}^s \langle z_0, e_{j,V}\rangle \langle e_{k,V}, z_0\rangle \langle Qe_{k,V}, e_{j,V}\rangle e^{i(\lambda_{k,V}-\lambda_{j,V})\tau} \\
&+ \alpha \sum_{j=2,k=1}^{\infty} \lambda_{j,V}^s \langle e_{j,V}, z_0\rangle \langle z_0, e_{k,V}\rangle \langle Qe_{k,V}, e_{j,V}\rangle e^{-i(\lambda_{k,V}-\lambda_{j,V})\tau} \\
&+ \sum_{k=2}^{\infty} \langle z_0, e_{1,V}\rangle \langle e_{k,V}, z_0\rangle \langle Qe_{k,V}, e_{1,V}\rangle e^{i(\lambda_{k,V}-\lambda_{1,V})\tau} \\
&- \sum_{k=2}^{\infty} \langle e_{1,V}, z_0\rangle \langle z_0, e_{k,V}\rangle \langle Qe_{k,V}, e_{1,V}\rangle e^{-i(\lambda_{k,V}-\lambda_{1,V})\tau} \\
=& \sum_{j=2,k=2}^{\infty} P(z_0, Q, j, k) e^{-i(\lambda_{j,V}-\lambda_{k,V})t} \\
&+ \sum_{j=2}^{\infty} \big(\alpha\lambda_{j,V}^s + 1\big) \langle z_0, e_{1,V}\rangle \langle e_{j,V}, z_0\rangle \langle Qe_{j,V}, e_{1,V}\rangle e^{-i(\lambda_{1,V}-\lambda_{j,V})t} \\
&- \sum_{j=2}^{\infty} \big(\alpha\lambda_{j,V}^s + 1\big) \langle e_{1,V}, z_0\rangle \langle z_0, e_{j,V}\rangle \langle Qe_{j,V}, e_{1,V}\rangle e^{i(\lambda_{1,V}-\lambda_{j,V})t}.
\end{aligned}
\tag{2.13}
$$

Here $P(z_0, Q, j, k)$ are constants. Define the set

$$J := \big\{\alpha \in \mathbb{R}\colon \alpha\lambda_{j,V}^s = -1 \text{ for some } j \geqslant 1\big\},$$

and take any $\alpha \notin J$. As $\langle z_0, e_{j,V}\rangle \neq 0$ for $j = 1, p$, Condition 2.2 and Lemma 3.10 in [28] imply that there is $T > 0$ such that $\Phi(T) \neq 0$. Thus there is a control $w \in C_0^\infty((0,T), \mathbb{R})$, $\|w\|_{C^k([0,T])} < d$ such that

$$\frac{\mathrm{d}\mathcal{V}(\mathcal{U}_T(z_0, \sigma w))}{\mathrm{d}\sigma}\bigg|_{\sigma=0} = \int_0^T \Phi(\tau) w(\tau)\,\mathrm{d}\tau \neq 0.$$

This implies that there is $\sigma_0 \in \mathbb{R}$ close to zero such that

$$\mathcal{V}\big(\mathcal{U}_T(z_0, \sigma_0 w)\big) < \mathcal{V}\big(\mathcal{U}_T(z_0, 0)\big) = \mathcal{V}(z_0),$$

which completes the proof.

Remark 2.7. Modifying slightly Condition 2.2, Theorem 2.3 can be restated for the eigenfunction $e_{i,V}$, $i \geqslant 1$. Indeed, one should replace $\lambda_{1,V}$ and $e_{1,V}$ by $\lambda_{i,V}$ and $e_{i,V}$ in Condition 2.2 and use the Lyapunov function

$$\mathcal{V}_i(z) := \alpha \big\|(-\Delta + V)^{\frac{s}{2}} P_{i,V} z\big\|^2 + 1 - \big|\langle z, e_{i,V}\rangle\big|^2, \quad z \in S \cap H^s_{(V)},$$

where $P_{i,V}$ is the orthogonal projection in L^2 onto the closure of the vector span of $\{e_{k,V}\}_{k\neq i}$.

2.3. Proof of Theorem 2.4

Let us look for a control in the form $\sigma + u(t)$, where $\sigma \in \mathbb{R}\backslash\{0\}$ is a constant. Then (1.1) takes the form

$$i\dot{z} = -\Delta z + \big(V(x) + \sigma Q(x)\big)z + u(t)Q(x)z,$$

and the idea of the proof is to show that the set of dipolar moments Q such that Condition 2.2 holds for V, Q replaced by $V + \sigma Q$, Q is residual. The proof is divided into three steps.

Step 1. First let us show that the set $\mathcal{Q}$ of all functions W verifying

$$\lambda_{1,W} - \lambda_{j,W} \neq \lambda_{p,W} - \lambda_{q,W} \tag{2.14}$$

for all integers $j, p, q \geqslant 1$ such that $\{1, j\} \neq \{p, q\}$ and $j \neq 1$ is residual in $C^\infty(\overline{D}, \mathbb{R})$. Fix $j, p, q \geqslant 1$ and denote by $\mathcal{Q}_{j,p,q}$ the set of functions $W \in C^\infty(\overline{D}, \mathbb{R})$ satisfying (2.14). As $\mathcal{Q} = \bigcap_{j,p,q} \mathcal{Q}_{j,p,q}$, it suffices to show that $\mathcal{Q}_{j,p,q}$ is open and dense. Continuity of the eigenvalues $\lambda_{k,W}$ from $C^\infty(\overline{D}, \mathbb{R})$ to $\mathbb{R}$ (e.g., see Theorem 8.1.2, [18]) implies that $\mathcal{Q}_{j,p,q}$ is open. Let us show that $\mathcal{Q}_{j,p,q}$ is dense in $C^\infty(\overline{D}, \mathbb{R})$. Indeed, by [2], the set $\widetilde{\mathcal{Q}}$ of functions $W \in C^\infty(\overline{D}, \mathbb{R})$ such that the spectrum of the operator $-\Delta + W$ is non-degenerate is residual in $C^\infty(\overline{D}, \mathbb{R})$. Take any $W \in \widetilde{\mathcal{Q}}$ and $P \in C^\infty(\overline{D}, \mathbb{R})$. It is well known that $\lambda_{k,W+\sigma P}$ and $e_{k,W+\sigma P}$ are analytic in σ in the neighborhood of 0 in $\mathbb{R}$ (e.g., see Theorem XII.8, [33]). Differentiating the identity

$$(-\Delta + W + \sigma P - \lambda_{k,W+\sigma P}) e_{k,W+\sigma P} = 0$$

with respect to σ at $\sigma = 0$, we get

$$(-\Delta + W - \lambda_{k,W}) \frac{\mathrm{d}e_{k,W+\sigma P}}{\mathrm{d}\sigma}\bigg|_{\sigma=0} + \left(P - \frac{\mathrm{d}\lambda_{k,W+\sigma P}}{\mathrm{d}\sigma}\bigg|_{\sigma=0}\right) e_{k,W} = 0.$$

Taking the scalar product of this identity with $e_{k,W}$, we obtain

$$\frac{\mathrm{d}\lambda_{k,W+\sigma P}}{\mathrm{d}\sigma}\bigg|_{\sigma=0} = \langle P, e_{k,W}^2 \rangle. \tag{2.15}$$

Thus

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}(\lambda_{1,W+\sigma P} - \lambda_{j,W+\sigma P} - \lambda_{p,W+\sigma P} + \lambda_{q,W+\sigma P})\bigg|_{\sigma=0} = \langle P, e_{1,W}^2 - e_{j,W}^2 - e_{p,W}^2 + e_{q,W}^2 \rangle. \tag{2.16}$$

By Theorem 4.1, the set $\mathcal{A}$ of all functions W such that the system $\{e_{j,W}^2\}_{j=1}^\infty$ is rationally independent is residual in $C^\infty(\overline{D}, \mathbb{R})$. Hence the set $\widetilde{\mathcal{Q}} \cap \mathcal{A}$ is residual. Fix any $W \in \widetilde{\mathcal{Q}} \cap \mathcal{A}$ and take $P \in C^\infty(\overline{D}, \mathbb{R})$ such that

$$\langle P, e_{1,W}^2 - e_{j,W}^2 - e_{p,W}^2 + e_{q,W}^2 \rangle \neq 0.$$

Then (2.16) implies that $W + \sigma P \in \mathcal{Q}_{j,p,q}$ for any σ sufficiently close to 0. This proves that $\mathcal{Q}_{j,p,q}$ is dense in $C^\infty(\overline{D}, \mathbb{R})$.

Step 2. Here we reduce the proof of the controllability of system (1.1), (1.2) with functions V and Q to that of with $V + \sigma Q$ and Q.

First let us suppose that $V \in C^\infty(\overline{D}, \mathbb{R})$ is such that the spectrum of the operator $-\Delta + V$ is non-degenerate. Take any sequence $\sigma_n \to 0$, $\sigma_n \neq 0$. Then the set $\mathcal{P}_1$ of all functions $Q \in C^\infty(\overline{D}, \mathbb{R})$ such that $V + \sigma_n Q \in \mathcal{Q}$ for all $n \geqslant 1$ is residual as a countable intersection of residual sets. On the other hand, the set $\mathcal{P}_2$ of functions $Q \in C^\infty(\overline{D}, \mathbb{R})$ such that $\langle Q e_{1,V}, e_{j,V} \rangle \neq 0$ for all $j \geqslant 2$ is also residual (see Section 3.4 in [28]). Define $\mathcal{P} := \mathcal{P}_1 \cap \mathcal{P}_2$. Let us show that problem (1.1), (1.2) is approximately controllable to $e_{1,V}$ for any $Q \in \mathcal{P}$. Fix an integer $n \geqslant 1$ and consider the problem

$$i\dot{z} = -\Delta z + V(x)z + \sigma_n Q(x)z + u(t)Q(x)z, \quad x \in D, \tag{2.17}$$

$$z|_{\partial D} = 0, \tag{2.18}$$

$$z(0, x) = z_0(x). \tag{2.19}$$

Let $\mathcal{U}_t^n(\cdot, u)$ be the resolving operator for problem (2.17), (2.18). Then we have $\mathcal{U}_t^n(\cdot, u) = \mathcal{U}_t(\cdot, u + \sigma_n)$. Notice that we cannot apply Theorem 2.3 with functions $V(x) + \sigma_n Q(x)$ and Q. Indeed, Condition 2.2, (i) is not necessarily satisfied. However, as $\sigma_n \to 0$, $Q \in \mathcal{P}_2$ and the spectrum of $-\Delta + V$ is non-degenerate, for any $N \geqslant 1$ there is $n^* \geqslant 1$ such that $\langle Q e_{1,V+\sigma_n Q}, e_{j,V+\sigma_n Q} \rangle \neq 0$, $j = 1, \ldots, N$ and $n \geqslant n^*$. Modifying slightly the arguments of the proof of Theorem 2.3, we show that this property is enough to conclude the approximate controllability. We need the following result.

Lemma 2.8. *Fix any constants $s > 0$, $d > 0$ and $\delta > 0$, an integer $k \geqslant 1$ and functions $Q \in \mathcal{P}$ and $V \in C^\infty(\overline{D}, \mathbb{R})$ such that the spectrum of the operator $-\Delta + V$ is non-degenerate. For any $M > 0$ and $\varepsilon > 0$ there is an integer $\hat{n} \geqslant 1$ such that for any $n \geqslant \hat{n}$ and $z_0 \in S \cap H^s_{(V+\sigma_n Q)}$ with $\langle z_0, e_{1,V+\sigma_n Q}\rangle \neq 0$ and $\|z_0\|_s < M$ we have*

$$\big\|\mathcal{U}^n_T(z_0, u) - e_{1,V+\sigma_n Q}\big\|_{s-\delta} < \varepsilon$$

for some time $T > 0$ and control $u \in C_0^\infty((0, T), \mathbb{R})$, $\|u\|_{C^k([0,T])} < d$.

To prove Theorem 2.4, let $z_0 \in S \cap H^s_{(V)}$ be such that $\|z_0\|_s < M$ and $\langle z_0, e_{1,V}\rangle \neq 0$. As z_0 is not necessarily in $H^s_{(V+\sigma_n Q)}$, we cannot apply Lemma 2.8. Take any $v \in C^\infty([0, \eta], \mathbb{R})$ such that $\frac{\mathrm{d}^k v}{\mathrm{d}t^k}(0) = \frac{\mathrm{d}^k v}{\mathrm{d}t^k}(\eta) = 0$ for all $k \geqslant 1$ and $v(0) = 0$. By Lemma 2.1, we have $y := \mathcal{U}_\eta(z_0, v) \in H^s_{(V+v(\eta)Q)}$. If $k \geqslant 1$ is sufficiently large and $\|v\|_{C^k([0,\eta])}$ is sufficiently small, then $\|y\|_s < M$ and $\langle y, e_{1,V+v(\eta)Q}\rangle \neq 0$. We can choose v such that $v(\eta) = \sigma_n$ for some $n \geqslant \hat{n}$. Applying Lemma 2.8 for the initial data y, we see that there is a time $\tilde{T} > 0$ and a control $\tilde{u}$ such that $\tilde{u} - v(\eta) \in C_0^\infty((0, \tilde{T}), \mathbb{R}))$ and

$$\big\|\mathcal{U}_{\tilde{T}}(y, \tilde{u}) - e_{1,V+v(\eta)Q}\big\|_{s-\delta} < \frac{\varepsilon}{2}.$$

For sufficiently small $\|v\|_{C^k([0,\eta])} + \eta$ we have

$$\big\|\mathcal{U}_\eta\big(\mathcal{U}_{\tilde{T}}(y, \tilde{u}), v(\eta - \cdot)\big) - e_{1,V}\big\|_{s-\delta} < \varepsilon.$$

This proves Theorem 2.4, when the spectrum of the operator $-\Delta + V$ is non-degenerate.

To prove the theorem for any function $V \in C^\infty(\overline{D}, \mathbb{R})$, notice that, Theorem 2.4 holds for system (1.1), (1.2) with functions $V + \sigma_n Q$ and Q for any $Q \in \mathcal{P}$ and $n \geqslant 1$. Indeed, from the construction of the set $\mathcal{P}$ it follows that the spectrum of $-\Delta + V + \sigma_n Q$ is non-degenerate. Repeating literally the arguments of the previous paragraph with Lemma 2.8 replaced by Theorem 2.4 for functions $V + \sigma_n Q$ and Q, we complete the proof.

Step 3. To prove Lemma 2.8, let $\mathcal{V}$ be defined by (2.1) with $V + \sigma_n Q$ instead of V. As $\langle z_0, e_{1,V+\sigma_n Q}\rangle \neq 0$, we can choose $\alpha > 0$ so small that $\mathcal{V}(z_0) < 1$. Notice that if $\mathcal{V}(z) < 1$, $z \in S$ then $\langle z, e_{1,V+\sigma_n Q}\rangle \neq 0$. On the other hand, for any $\varepsilon > 0$ there is an integer $N \geqslant 1$ such that if $\|z\|_s \leqslant M$, $z \in S \cap H^s_{(V+\sigma_n Q)}$ and $\langle z, e_{j,V+\sigma_n Q}\rangle = 0$ for any $j \in [2, N]$, then $\|z - ce_{1,V+\sigma_n Q}\|_{s-\delta} < \varepsilon$ for some $c := c(z) \in \mathbb{C}$ with $|c| = 1$.

We need the following lemma.

Lemma 2.9. *There is a finite or countable set $J \subset \mathbb{R}^*_+$ and an integer $\hat{n} \geqslant 1$ such that for any $\alpha \notin J$, $n \geqslant \hat{n}$ and $z \in H^s_{(V+\sigma_n Q)}$ with $\langle z, e_{1,V+\sigma_n Q}\rangle \neq 0$ and $\langle z, e_{j,V+\sigma_n Q}\rangle \neq 0$ for some integer $j \in [2, N]$, there is a time $T > 0$ and a control $u \in C_0^\infty((0, T), \mathbb{R})$ verifying*

$$\mathcal{V}\big(\mathcal{U}^n_T(z, u)\big) < \mathcal{V}(z).$$

Proof. As $Q \in \mathcal{P}$, for sufficiently large $\hat{n}$ we have $\langle Qe_{1,V+\sigma_n Q}, e_{j,V+\sigma_n Q}\rangle \neq 0$ for $j = 2, \ldots, N$ and $n \geqslant \hat{n}$. Repeating the arguments of the proof of Proposition 2.6, one should just notice that if sum (2.13) equals zero for all $\tau \geqslant 0$, then $\langle z_0, e_{j,V+\sigma_n Q}\rangle = 0$ for all $j \in [2, N]$. This contradicts the hypothesis of the lemma. □

As in the proof of Theorem 2.3, we define the set

$$\mathcal{K} = \big\{z \in H^s_{(V+\sigma_n Q)}\colon \mathcal{U}^n_{T_\ell}(z_0, u_\ell) \to z \text{ in } H^{s-\delta} \text{ for some } T_\ell \geqslant 0,$$
$$u_\ell \in C_0^\infty\big((0, T_\ell), \mathbb{R}\big), \|u_\ell\|_{C^k([0,T_\ell])} < d \text{ and for any } \delta > 0\big\}.$$

Let

$$m := \inf_{z \in \mathcal{K}} \mathcal{V}(z).$$

This infimum is attained at some point $e \in \mathcal{K}$. Let us show that

$$\|e - ce_{1,V+\sigma_n Q}\|_{s-\delta} < \varepsilon$$

for some $c := c(z) \in \mathbb{C}$ with $|c| = 1$. Indeed, it follows from the choice of α that $\mathcal{V}(e) \leqslant \mathcal{V}(z_0) < 1$. Hence $\langle e, e_{1,V+\sigma_n Q}\rangle \neq 0$. Suppose that $\langle e, e_{j,V+\sigma_n Q}\rangle \neq 0$ for some integer $j \in [2, N]$. By Lemma 2.9, there is a time $T > 0$ and a control $u \in C_0^\infty((0, T), \mathbb{R})$ such that

$$\mathcal{V}\big(\mathcal{U}_T^n(e, u)\big) < \mathcal{V}(e). \tag{2.20}$$

Define $\tilde{u}_\ell(t) = u_\ell(t)$, $t \in [0, T_\ell]$ and $\tilde{u}_\ell(t) = u(t)$, $t \in [T_\ell, T_\ell + T]$. Then $\tilde{u}_\ell \in C_0^\infty((0, T_\ell + T), \mathbb{R})$ and

$$\mathcal{U}_{T_\ell+T}^n(z_0, \tilde{u}_\ell) \to \mathcal{U}_T^n(e, u) \quad \text{in } H^{s-\delta}.$$

This implies that $\mathcal{U}_T^n(e, u) \in \mathcal{K}$. Clearly, (2.20) contradicts the definition of e. It follows that $\langle e, e_{j,V+\sigma_n Q}\rangle = 0$ for any $j \in [2, N]$, hence $\|e - ce_{1,V+\sigma_n Q}\|_{s-\delta} < \varepsilon$ for some $c := c(e) \in \mathbb{C}$, $|c| = 1$. Without loss of generality, we can suppose that $c = 1$.

3. Applications

3.1. Global exact controllability

The following controllability result for system (1.1), (1.2) is obtained by Beauchard [8] in the case $m = 1$ and $V = 0$.

Theorem 3.1. *There is a residual set $\mathcal{Q}'$ in $W^{3,\infty}((0, 1), \mathbb{R})$ such that for any $Q \in \mathcal{Q}'$ and some constants $T > 0$ and $\varepsilon > 0$ the following exact controllability property holds: for any $z_0, z_1 \in S \cap H_{(0)}^5$ with*

$$\|z_i - e_{1,0}\|_5 < \varepsilon, \quad i = 1, 2$$

there is a control $u \in H_0^1([0, T], \mathbb{R})$ such that

$$\mathcal{U}_T(z_0, u) = z_1.$$

On the other hand, by Theorem 2.4, problem (1.1), (1.2) with $m = 1$ and $V = 0$ is approximately controllable to $e_{1,0}$ in $H^{5+\delta}$ generically with respect to Q in $C^\infty([0, 1], \mathbb{R})$. Literally the same proof shows that in the case of the phase space $H^{5+\delta}$ the approximate controllability property holds generically with respect to Q in $W^{3,\infty}((0, 1), \mathbb{R})$. Thus, combining Theorems 2.4 and 3.1, we obtain.

Theorem 3.2. *There is a residual set $\widehat{\mathcal{Q}}$ in $W^{3,\infty}((0, 1), \mathbb{R})$ such that for any $Q \in \widehat{\mathcal{Q}}$ and any $z_0, z_1 \in S \cap H_{(0)}^{5+\delta}$ there is a time $T > 0$ and a control $u \in H_0^1([0, T], \mathbb{R})$ verifying*

$$\mathcal{U}_T(z_0, u) = z_1.$$

Remark 3.3. It is shown in [7,11] that if we take $Q(x) = x$, then for any $N \geqslant 1$ there is a constant $\sigma^* > 0$ such that for any $\sigma \in (0, \sigma^*)$ we have

(i) $\langle xe_{1,\sigma x}, e_{j,\sigma x}\rangle \neq 0$ for all $j \geqslant 2$,
(ii) $\lambda_{1,\sigma x} - \lambda_{j,\sigma x} \neq \lambda_{p,\sigma x} - \lambda_{q,\sigma x}$ for all $j, p, q \geqslant 1$ such that $\{1, j\} \neq \{p, q\}$ and $2 \leqslant j \leqslant N$.

These properties imply that the proof of Theorem 2.4 works and the system with $Q(x) = x$ is approximately controllable. Indeed, properties (i) and (ii) are sufficient to conclude that in the proof of Lemma 2.9 sum (2.13) is not equal to zero for all $\tau \geqslant 0$. This is the only place, where the hypotheses on the dipolar moment Q are used.

On the other hand, by Beauchard [7], for $Q(x) = x$ the problem is exactly controllable in a neighborhood of $e_{1,0}$. Thus global exact controllability property holds for $Q(x) = x$.

3.2. Uniqueness of stationary measure

Let us consider the Schrödinger equation with a potential which has a random time-dependent amplitude:

$$i\dot{z} = -\Delta z + V(x)z + \beta(t)Q(x)z, \quad x \in D, \tag{3.1}$$

$$z|_{\partial D} = 0, \tag{3.2}$$

$$z(0) = z_0, \tag{3.3}$$

where $V, Q \in C^\infty(\overline{D}, \mathbb{R})$ are given functions. We assume that $\beta(t)$ is a random process of the form

$$\beta(t) = \sum_{k=0}^{+\infty} I_k(t)\eta_k(t-k), \quad t \geqslant 0, \tag{3.4}$$

where $I_k(\cdot)$ is the indicator function of the interval $[k, k+1)$ and η_k are independent identically distributed random variables in $L^2([0,1], \mathbb{R})$. Then $\mathcal{U}_k(\cdot, \beta)$ is a homogeneous Markov chain with respect to the filtration $\mathcal{F}_k$ generated by $\eta_0, \ldots, \eta_{k-1}$ (e.g., see [29]). For any $z \in S$ and $\Gamma \in \mathcal{B}(S)$, the transition functions corresponding to the process $\mathcal{U}_k(\cdot, \beta)$ are defined by $P_k(z, \Gamma) = \mathbb{P}\{\mathcal{U}_k(z, \beta) \in \Gamma\}$ and the Markov operators by

$$\mathfrak{P}_k f(z) = \int_S P_k(z, \mathrm{d}v) f(v), \qquad \mathfrak{P}_k^* \mu(\Gamma) = \int_S P_k(v, \Gamma)\mu(\mathrm{d}v),$$

where $f \in C_b(S)$ and $\mu \in \mathcal{P}(S)$. Let us recall that a measure $\mu \in \mathcal{P}(S)$ is stationary for (3.1), (3.2), (3.4) if $\mathfrak{P}_1^*\mu = \mu$. The question of existence of a stationary measure is an open problem. In this section, we derive the uniqueness from Theorem 2.3. We need the following condition.

Condition 3.4. The random variables η_k have the form

$$\eta_k(t) = \sum_{j=1}^{\infty} b_j \xi_{jk} g_j(t), \quad t \in [0, 1],$$

where $\{g_j\}$ is an orthonormal basis in $L^2([0,1], \mathbb{R})$, $b_j > 0$ are constants with

$$\sum_{j=1}^{\infty} b_j^2 < \infty,$$

and ξ_{jk} are independent real-valued random variables such that $\mathbb{E}\xi_{jk}^2 = 1$. Moreover, the distribution of ξ_{jk} possesses a continuous density ρ_j with respect to the Lebesgue measure and $\rho_j(r) > 0$ for all $r \in \mathbb{R}$.

Theorem 3.5. *Under Conditions* 2.2 *and* 3.4*, problem* (3.1), (3.2), (3.4) *has at most one stationary measure on S.*

This theorem is derived from the following general result (cf. [22]). Let X be a Polish space and let $P_k(z, \Gamma)$ be a Markov transition function satisfying the Feller property. We denote by $\mathfrak{P}_k$ and $\mathfrak{P}_k^*$ the corresponding Markov semigroups. Recall that a stationary measure μ for $\mathfrak{P}_k^*$ is said to be ergodic if

$$\sigma_n(f) := \frac{1}{n}\sum_{i=0}^{n-1} \mathfrak{P}_i f(z) \to (f, \mu) \tag{3.5}$$

for any $f \in C_b(X)$ and for μ-a.e. $z \in X$, where $(f, \mu) = \int_X f(z)\mu(\mathrm{d}z)$.

Theorem 3.6. *Suppose that P_k satisfies the following two conditions.*

(i) *For any $f \in \mathcal{L}(X)$ there is a constant $L_f > 0$ such that $\mathfrak{P}_k f$ is L_f-Lipschitz for any $k \geqslant 0$.*
(ii) *For any point $z \in X$ and any ball $B \subset X$ there is $l \geqslant 1$ such that $P_l(z, B) > 0$.*

Then $\mathfrak{P}_k^*$ has at most one stationary distribution.

Proof of Theorem 3.5. Let us show that properties (i) and (ii) are verified for system (3.1), (3.2), (3.4). Take any function $f \in \mathcal{L}(S)$. Then

$$\begin{aligned}\left|\mathfrak{P}_k f(z_1) - \mathfrak{P}_k f(z_2)\right| &= \left|\mathbb{E}\big(f\big(\mathcal{U}_k(z_1,\beta)\big) - f\big(\mathcal{U}_k(z_2,\beta)\big)\big)\right| \\ &\leqslant \|f\|_{\mathcal{L}}\mathbb{E}\big\|\mathcal{U}_k(z_1,\beta) - \mathcal{U}_k(z_2,\beta)\big\| = \|f\|_{\mathcal{L}}\|z_1 - z_2\|,\end{aligned}$$

which implies (i). To show (ii), notice that Condition 3.4 implies that

$$\mathbb{P}\big\{\|u-\beta\|_{L^2([0,l])} < \varepsilon\big\} > 0$$

for any $u \in L^2([0,l])$ and $\varepsilon > 0$. Moreover, using the continuity of the mapping $\mathcal{U}_l(z_0,\cdot) : L^2([0,l]) \to L^2(D)$, for any $\delta > 0$ we can find a constant $\varepsilon > 0$ such that

$$\mathbb{P}\big\{\big\|\mathcal{U}_l(z_0,\beta) - \mathcal{U}_l(z_0,u)\big\| < \delta\big\} \geqslant \mathbb{P}\big\{\|u-\beta\|_{L^2([0,l])} < \varepsilon\big\} > 0.$$

Hence, any point $\mathcal{U}_l(z_0,u)$, $u \in L^2([0,l])$ is in the support of the measure $\mathcal{D}(\mathcal{U}_l(z_0,\beta)) = P_l(z_0,\cdot)$. By Theorem 2.3, problem (1.1), (1.2) is approximately controllable in S (cf. Theorem 3.5 in [28]), hence the set $\{\mathcal{U}_l(z_0,u)$: $u \in L^2([0,l]), l \geqslant 0\}$ is dense in S. This implies (ii). Applying Theorem 3.6, we complete the proof. □

Proof of Theorem 3.6. In view of ergodic decomposition of stationary distributions (e.g., see [21]), it suffices to prove that there is at most one ergodic stationary measure. Let μ_1 and μ_2 be two ergodic stationary measures. Suppose there is a function $f \in \mathcal{L}(X)$ such that $(f,\mu_1) \neq (f,\mu_2)$. Let X_i, $i = 1,2$, be the set of convergence in (i) with $\mu = \mu_i$. Then $\mu_i(X_i) = 1$ and $X_1 \cap X_2 = \emptyset$. Furthermore, in view of condition (ii), for any ball $B \subset X$ there is $l \geqslant 1$ such that

$$\mu_i(B) = \int_X P_l(z,B)\mu_i(\mathrm{d}z) > 0.$$

Thus $\operatorname{supp}\mu_i = X$, and therefore $\overline{X_i} = X$. Now let $K_n \subset X$ be an increasing sequence of compact subsets such that $\mu_i(K_n) > 1 - 2^{-n}$. Then, by condition (i) and the Arzelà theorem, there is a subsequence $k_j \to \infty$ such that for any $n \geqslant 1$ the sequence $\sigma_{k_j}(f)$ converges uniformly on K_n to an L_f-Lipschitz bounded function $\overline{f}_n$. Let us set $Y = \bigcup_n K_n$ and define an L_f-Lipschitz function $\overline{f} : Y \to \mathbb{R}$ such that $\overline{f}|_{K_n} = \overline{f}_n$. Since $\mu_i(Y) = 1$, we see that $\mu_i(Y \cap X_i) = 1$ and $\overline{Y \cap X_i} = X$, where we used again condition (ii). We conclude that $\overline{f}$ must be a constant function on X equal to (f,μ_i). The contradiction obtained shows that $\mu_1 = \mu_2$. □

4. Independence of squares of eigenfunctions

Recall that the functions $\{f_j\}_{j=1}^\infty \subset C(D)$ are said to be rationally independent, if for any $N \geqslant 1$ and $\alpha_k \in \mathbb{Q}$, $k = 1, \ldots, N$ with $|\alpha_1| + \cdots + |\alpha_N| > 0$ we have

$$\sum_{k=1}^N \alpha_k f_k \not\equiv 0.$$

Theorem 4.1. *The set $\mathcal{A}$ of all functions $Q \in C^\infty(\overline{D},\mathbb{R})$ such that the system $\{e_{j,Q}^2\}_{j=1}^\infty$ is rationally independent is residual in $C^\infty(\overline{D},\mathbb{R})$.*

Proof. The proof of this theorem is inspired by the paper [31] by Privat and Sigalotti, where the linear independence of the squares of the eigenfunctions of the Dirichlet Laplacian is established to hold generically with respect to the domain D.

Take any $N \geqslant 1$ and $\alpha_k \in \mathbb{Q}$, $k = 1, \ldots, N$, with $|\alpha_1| + \cdots + |\alpha_N| > 0$. It suffices to show that the set $\mathcal{A}_{\alpha,N}$ of all functions $Q \in C^\infty(\overline{D},\mathbb{R})$ such that the eigenvalues $\lambda_{j,Q}$, $j = 1, \ldots, N$, are simple and

$$\sum_{k=1}^N \alpha_k e_{k,Q}^2 \not\equiv 0$$

is open and dense in $C^\infty(\overline{D},\mathbb{R})$. Indeed, noting that

$$\mathcal{A}\supset\bigcap_{N\geqslant 1,\alpha\in\mathbb{Q}^N}\mathcal{A}_{\alpha,N}$$

we complete the proof. The fact that $\mathcal{A}_{\alpha,N}$ is open follows from the continuity of $\lambda_{j,Q}$ and $e_{k,Q}$ with respect to $Q\in C^\infty(\overline{D},\mathbb{R})$ at any $Q_0\in\mathcal{A}_{\alpha,N}$ (e.g., see [20]).

To prove that $\mathcal{A}_{\alpha,N}$ is dense, we first show that the operator $-\Delta+Q$ satisfies the hypothesis of Theorem B in [34] for any $Q\in C(\overline{D},\mathbb{R})$. This implies that any functions $Q_0,Q_1\in C(\overline{D},\mathbb{R})$ can be connected by an analytic curve $Q_s\in C(\overline{D},\mathbb{R})$, $s\in[0,1]$ such that the spectrum of $-\Delta+Q_s$ is simple for any $s\in(0,1)$. In particular, λ_{k,Q_s} and e_{k,Q_s} are analytic in $s\in(0,1)$. Then we show that $\mathcal{A}_{\alpha,N}$ is non-empty. Taking any $Q_1\in\mathcal{A}_{\alpha,N}$, we see that, by analyticity, also $Q_s\in\mathcal{A}_{\alpha,N}$ for all $s\in[0,1]\backslash I$, where $I\subset[0,1]$ is an at most countable set. Thus $Q_{s_n}\to Q_0$ and $Q_{s_n}\in\mathcal{A}_{\alpha,N}$ for any $s_n\to 0$ such that $s_n\in[0,1]\backslash I$.

Step 1. The family $-\Delta+Q$, $Q\in C(\overline{D},\mathbb{R})$ satisfies the hypothesis of Theorem B in [34]. Indeed, the function Q is in the separable Banach space $C(\overline{D},\mathbb{R})$, and the operator $-\Delta+Q$ is self-adjoint in $L^2(D)$ with spectrum which is discrete, of finite multiplicity, and without finite accumulation points. Thus it remains to show that the condition SAH2 in [34] is also verified. Notice that for all $Q,P\in C(\overline{D},\mathbb{R})$ and $\varepsilon\in\mathbb{R}$ we have

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}(-\Delta+Q+\varepsilon P)=P.$$

Hence we have to prove that for any eigenvalue λ of $-\Delta+Q$ of multiplicity $n\geqslant 2$ there exist two orthonormal eigenfunctions v_1 and v_2 corresponding to λ such that the functionals $P\to\langle P,v_1^2-v_2^2\rangle$ and $P\to\langle P,v_1v_2\rangle$ are linearly independent. Suppose, by contradiction, that for some eigenfunctions v_1 and v_2 we have

$$v_1^2(x)-v_2^2(x)-cv_1(x)v_2(x)=0\quad\text{for all } x\in D,$$

where $c\in\mathbb{R}$ is a constant. Thus

$$\left(v_1-\frac{c}{2}v_2\right)^2=\frac{c^2+4}{4}v_2^2,$$

where $v_1-\frac{c}{2}v_2$ and $\frac{\sqrt{c^2+4}}{2}v_2$ are linearly independent eigenfunctions of $-\Delta+Q$ corresponding to the eigenvalue λ. Combining this with the unique continuation theorem for the operator $-\Delta+Q$ (see [19]), we get that $v_1-\frac{c}{2}v_2=\pm\frac{\sqrt{c^2+4}}{2}v_2$, which contradicts the fact that v_1 and v_2 are linearly independent. Thus the functionals $\langle P,v_1^2-v_2^2\rangle$ and $\langle P,v_1v_2\rangle$ are linearly independent. Applying Theorem B in [34], we see that any $Q_0,Q_1\in C(\overline{D},\mathbb{R})$ can be connected by an analytic curve $Q_s\in C(\overline{D},\mathbb{R})$, $s\in[0,1]$ such that the spectrum of $-\Delta+Q_s$ is simple for any $s\in(0,1)$.

Step 2. To show that $\mathcal{A}_{\alpha,N}$ is non-empty, we use the following result from the inverse spectral theory for Sturm–Liouville problems.

Theorem 4.2. *Let $\{\lambda_k\}_{k=1}^N$ and $\{\lambda'_k\}_{k=1}^N$ be two sets of positive constants such that*

$$\lambda_1<\lambda'_1<\lambda_2<\lambda'_2<\cdots. \tag{4.1}$$

Then for any $a>0$ and $n\geqslant 1$ there is a function $W\in L^2([-2^na,2^na],\mathbb{R})$ such that

$$\lambda_k=\lambda_{k,W}^{(-2^na,2^na)},\quad \lambda'_k=\lambda_{k,W}^{(0,a)},\quad k=1,\dots,N. \tag{4.2}$$

This theorem is a consequence of a much stronger result by Pivovarchik (see Theorem 2.1 in [30]).

Without loss of generality, we can assume that $0\in D$. Let us choose $a>0$ and $n\geqslant 1$ such that

$$B':=(0,a)^m\subset D\subset\left(-2^na,2^na\right)^m=:B. \tag{4.3}$$

By the min–max principle (e.g., see [33]) and (4.3), we have

$$\lambda_{k,Q}^B\leqslant\lambda_{k,Q}^D\leqslant\lambda_{k,Q}^{B'} \tag{4.4}$$

for any $Q \in L^2(B,\mathbb{R})$ and $k \geqslant 1$. Let us suppose that Q is of the form

$$Q(x_1,\dots,x_m) = P(x_1) + R(x_2) + \dots + R(x_m),$$

where $c > 0$ is a constant and $P, R \in L^2([-2^n a, 2^n a])$. Then the eigenvalues are of the form

$$\lambda_{k,Q}^{B'} = \lambda_{i_1,P}^{(0,a)} + \lambda_{i_2,R}^{(0,a)} + \dots + \lambda_{i_m,R}^{(0,a)},$$
$$\lambda_{k,Q}^{B} = \lambda_{j_1,P}^{(-2^n a,2^n a)} + \lambda_{j_2,R}^{(-2^n a,2^n a)} + \dots + \lambda_{j_m,R}^{(-2^n a,2^n a)}$$

for some integers $i_p, j_p \geqslant 1$, $p = 1,\dots,m$. Let $\{\lambda_k,\lambda_k'\}_{k=1}^N$ and $\{\tilde\lambda_k,\tilde\lambda_k'\}_{k=1}^N$ be two sets of positive constants verifying (4.1). Applying Theorem 4.2, let $P, R \in L^2([-2^n a, 2^n a],\mathbb{R})$ be such that (4.2) holds for $\{\lambda_k,\lambda_k'\}_{k=1}^N$ and $\{\tilde\lambda_k,\tilde\lambda_k'\}_{k=1}^N$, respectively. If $\tilde\lambda_2 > \lambda_N'$, then

$$\lambda_{k,Q}^{B'} = \lambda_{k,P}^{(0,a)} + (m-1)\lambda_{1,R}^{(0,a)} = \lambda_k' + (m-1)\tilde\lambda_1', \tag{4.5}$$
$$\lambda_{k,Q}^{B} = \lambda_{k,P}^{(-2^n a,2^n a)} + (m-1)\lambda_{1,R}^{(-2^n a,2^n a)} = \lambda_k + (m-1)\tilde\lambda_1 \tag{4.6}$$

for $k = 1,\dots,N$. We can choose $\{\lambda_k,\lambda_k'\}_{k=1}^N$ and $\{\tilde\lambda_k,\tilde\lambda_k'\}_{k=1}^N$ such that

$$\sum_{k=1}^N \alpha_k \mu_k \neq 0 \tag{4.7}$$

for all $\mu_k \in [\lambda_{k,Q}^B, \lambda_{k,Q}^{B'}]$. Indeed, we deduce from (4.5) and (4.6)

$$\begin{aligned}\left|\sum_{k=1}^N \alpha_k\mu_k - \sum_{k=1}^N \alpha_k\lambda_{k,Q}^{B'}\right| &\leqslant \sum_{k=1}^N |\alpha_k|\left|\mu_k - \lambda_{k,Q}^{B'}\right| \leqslant \sum_{k=1}^N |\alpha_k|\left(\lambda_{k,Q}^{B'} - \lambda_{k,Q}^{B}\right) \\ &\leqslant \Big(\sup_{k\in[1,N]}\big(\lambda_k' - \lambda_k\big) + (m-1)\big(\tilde\lambda_1' - \tilde\lambda_1\big)\Big)\sum_{k=1}^N |\alpha_k|. \end{aligned}\tag{4.8}$$

Take any $\varepsilon > 0$ and choose $\{\lambda_k,\lambda_k'\}_{k=1}^N$ and $\{\tilde\lambda_1,\tilde\lambda_1'\}$ such that

$$\Big(\sup_{k\in[1,N]}\big(\lambda_k' - \lambda_k\big) + (m-1)\big(\tilde\lambda_1' - \tilde\lambda_1\big)\Big)\sum_{k=1}^N |\alpha_k| < \varepsilon. \tag{4.9}$$

On the other hand, we can choose $\{\lambda_k'\}_{k=1}^N$ and $\{\tilde\lambda_k'\}_{k=1}^N$ such that

$$\left|\sum_{k=1}^N \alpha_k \lambda_{k,Q}^{B'}\right| > \varepsilon.$$

Combining this with (4.8) and (4.9), we arrive at (4.7). Thus (4.4) implies that

$$\sum_{k=1}^N \alpha_k \lambda_{k,Q}^D \neq 0. \tag{4.10}$$

Without loss of generality, we can assume that $Q \in C^\infty(\overline{D},\mathbb{R})$. Take any $\tilde Q \in C^\infty(\overline{D},\mathbb{R}) \setminus \mathcal{A}_{\alpha,N}$ (if $C^\infty(\overline{D},\mathbb{R}) = \mathcal{A}_{\alpha,N}$ then the proof is completed). By Step 2, the functions Q and $\tilde Q$ can be connected by an analytic curve $\tilde Q_s \in C(\overline{D},\mathbb{R})$, $s \in (0,1)$ such that the spectrum of $-\Delta + \tilde Q_s$ is simple for any $s \in (0,1)$. We deduce from (4.10) that the analytic function $\sum_{k=1}^N \alpha_k \lambda_{k,\tilde Q_s}^D$ is non-constant on $[0,1]$. This implies that

$$0 \neq \frac{\mathrm{d}}{\mathrm{d}s}\sum_{k=1}^N \alpha_k \lambda_{k,\tilde Q_s}^D = \left\langle \frac{\mathrm{d}\tilde Q_s}{\mathrm{d}s}, \sum_{k=1}^N \alpha_k \big(e_{k,\tilde Q_s}^D\big)^2 \right\rangle \tag{4.11}$$

for all $s \in [0,1]\backslash\tilde{I}$, where $\tilde{I} \subset [0,1]$ is an at most countable set (cf. Step 1 of the proof of Theorem 3.1). Indeed, taking the derivative of the identity

$$\left(-\Delta + \tilde{Q}_s - \lambda^D_{k,\tilde{Q}_s}\right)e^D_{k,\tilde{Q}_s} = 0$$

with respect to s, we get

$$\left(-\Delta + \tilde{Q}_s - \lambda^D_{k,\tilde{Q}_s}\right)\frac{\mathrm{d}e^D_{k,\tilde{Q}_s}}{\mathrm{d}s} + \left(\frac{\mathrm{d}\tilde{Q}_s}{\mathrm{d}s} - \frac{\mathrm{d}\lambda^D_{k,\tilde{Q}_s}}{\mathrm{d}s}\right)e^D_{k,\tilde{Q}_s} = 0.$$

Taking the scalar product of this identity with $e^D_{k,\tilde{Q}_s}$, we obtain

$$\frac{\mathrm{d}\lambda^D_{k,\tilde{Q}_s}}{\mathrm{d}s} = \left\langle \frac{\mathrm{d}\tilde{Q}_s}{\mathrm{d}s}, \left(e^D_{k,\tilde{Q}_s}\right)^2 \right\rangle, \tag{4.12}$$

which implies (4.11). Finally, (4.11) shows that

$$\sum_{k=1}^{N} \alpha_k \left(e^D_{k,\tilde{Q}_s}\right)^2 \neq 0$$

for all $s \in [0,1]\backslash\tilde{I}$ and $\tilde{Q}_s \in \mathcal{A}_{\alpha,N}$. Thus $\mathcal{A}_{\alpha,N}$ is non-empty. □

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