Available online at www.sciencedirect.com

SciVerse ScienceDirect

J. Math. Pures Appl. 97 (2012) 295–317

JOURNAL DE MATHÉMATIQUES PURES ET APPLIQUÉES

www.elsevier.com/locate/matpur

Global exact controllability in infinite time of Schrödinger equation

Vahagn Nersesyan [a,*], Hayk Nersisyan [b]

[a] *Laboratoire de Mathématiques, UMR CNRS 8100, Université de Versailles-Saint-Quentin-en-Yvelines, F-78035 Versailles, France*
[b] *Laboratoire de Mathématiques, UMR CNRS 8088, Université de Cergy-Pontoise, F-95000 Cergy-Pontoise, France*

Received 12 August 2010

Available online 25 November 2011

Abstract

In this paper, we study the problem of controllability of Schrödinger equation. We prove that the system is exactly controllable in infinite time to any position. The proof is based on an inverse mapping theorem for multivalued functions. We show also that the system is not exactly controllable in finite time in lower Sobolev spaces.

Résumé

Dans cet article, on étudie le problème de contrôlabilité pour l'équation de Schrödinger. Nous montrons que le système est exactement contrôlable en temps infini. La démonstration utilise un théorème d'inversion locale pour des multifonctions. On montre aussi que le système n'est pas exactement contrôlable en temps fini dans les espaces de Sobolev d'ordre inférieur.

Keywords: Schrödinger equation; Controllability; Multivalued mapping; Kolmogorov ε-entropy

1. Introduction

The paper is devoted to the study of the following controlled Schrödinger equation

$$i\dot{z} = -\Delta z + V(x)z + u(t)Q(x)z, \tag{1.1}$$

$$z|_{\partial D} = 0, \tag{1.2}$$

$$z(0,x) = z_0(x). \tag{1.3}$$

We assume that space variable x belongs to a rectangle $D \subset \mathbb{R}^d$, $V, Q \in C^\infty(\overline{D}, \mathbb{R})$ are given functions, u is the control, and z is the state. We prove that the linearization of this system is exactly controllable in Sobolev spaces in infinite time. Application of this result gives global exact controllability in infinite time in H^3 for $d = 1$. We show also that the system is not exactly controllable in finite time in lower Sobolev spaces.

Let us recall some previous results on the controllability problem of Schrödinger equation. In [6], Beauchard proves an exact controllability result for the system with $d = 1$, $D = (-1, 1)$ and $Q(x) = x$ in H^7-neighborhoods of

* Corresponding author.
E-mail addresses: Vahagn.Nersesyan@math.uvsq.fr (V. Nersesyan), Hayk.Nersisyan@u-cergy.fr (H. Nersisyan).

0021-7824/$ – see front matter
doi:10.1016/j.matpur.2011.11.005

the eigenfunctions. Beauchard and Coron [8] established later a partial global exact controllability result, showing that the system in question is also controlled between neighborhoods of eigenfunctions. Recently, Beauchard and Laurent [10] simplified the proof of [6] and generalized it to the case of the nonlinear equation. The proofs of [6,8,10] work also for the neighborhoods of finite linear combinations of eigenfunctions. In the case of infinite linear combinations, these arguments do not work, since the linearized system does not verify the property of spectral gap (even if the problem is 1-D), hence the Ingham inequality cannot be applied.

Chambrion et al. [12], Privat and Sigalotti [26], and Mason and Sigalotti [19] prove that (1.1), (1.2) is approximately controllable in L^2 generically with respect to function Q and domain D. In [23,22], the first author of this paper proves a stabilization result and a property of global approximate controllability to eigenstates for Schrödinger equation. Combination of these results with the local exact controllability property obtained by Beauchard [6] gives global exact controllability in finite time for $d = 1$ in the spaces $H^{3+\varepsilon}$, $\varepsilon > 0$. See also papers [27,30,3,2,1,9] for controllability of finite-dimensional systems and papers [17,18,5,31,13,20,15] for controllability properties of various Schrödinger systems.

In this article, we study the properties of control system on the time half-line $\mathbb{R}_+$ instead of a finite interval $[0, T]$, as in all above cited papers. We study the mapping, which associates to initial condition z_0 and control u the ω-limit set of the corresponding trajectory. We consider this mapping as a multivalued function in the phase space. We show that this multivalued function is differentiable with differential equal to the limit of the linearization of (1.1), (1.2), when time t goes to infinity. Observing that the linearized system is controllable in infinite time at almost any point, we conclude the controllability of the nonlinear system (in the case $d = 1$), using an inverse mapping theorem for multivalued functions [21] by Nachi and Penot. Thus (1.1), (1.2) is exactly controllable near any point in the phase space, hence globally. The controllability of the linearized system is proved for any $d \geqslant 1$, but this result is not directly applicable to the study of the nonlinear system with $d \geqslant 2$. We have a loss of regularity: the solution of the nonlinear problem exists for more regular controls than the ones used to control the linear problem. The multidimensional case is treated in our forthcoming paper.

To our knowledge, the inverse mapping theorem for multivalued functions was never used before in the theory of control of PDEs. Our proof does not rely on the particular asymptotics of the eigenvalues of Dirichlet Laplacian, so it is likely to work in other settings. Considering the problem in infinite time enables us to prove the controllability of the linearized system in the case of any space dimension $d \geqslant 1$, even when the gap condition is not verified for the eigenvalues (which is the case for $d \geqslant 3$).

In the second part of the paper, we study the problem of non-controllability for (1.1), (1.2) in finite time. We prove that the system is not exactly controllable in finite time in the spaces H^k with $k \in (0, d)$. Let us recall that previously Ball, Marsden and Slemrod [4] and Turinici [29] have shown that the problem is not controllable in the space H^2. Our result is inspired by the paper [28] of Shirikyan, where the non-controllability of 2D Euler equation is established. More precisely, it is proved in [28] that, if the Euler system is controlled by finite dimensional external force, then the set of all reachable points in a given time $T > 0$ cannot cover a ball in the phase space. Later this result was generalized by the second author of the present paper, in [24]: in the case of 3D Euler equation it is proved that the union of all sets of reachable points at all times $T > 0$ also does not cover a ball.

Using ideas of Shirikyan, we prove that the image by the resolving operator of a ball in the space of controls has a Kolmogorov ε-entropy strictly less than that of a ball in the phase space H^k. This implies the non-controllability.

Notation. In this paper, we use the following notation. Let

$$\ell^2 := \left\{ \{a_j\} \in \mathbb{C}^\infty \colon \ \big\|\{a_j\}\big\|_{\ell^2}^2 = \sum_{j=1}^{+\infty} |a_j|^2 < +\infty \right\},$$
$$\ell_0^2 := \big\{ \{a_j\} \in \ell^2 \colon \ a_1 \in \mathbb{R} \big\}.$$

We denote by $H^s := H^s(D)$ the Sobolev space of order $s \geqslant 0$. Consider the Schrödinger operator $-\Delta + V$, $V \in C^\infty(\overline{D}, \mathbb{R})$ with $\mathcal{D}(-\Delta + V) := H_0^1 \cap H^2$. Let $\{\lambda_{j,V}\}$ and $\{e_{j,V}\}$ be the sets of eigenvalues and normalized eigenfunctions of this operator. Let $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ be the scalar product and the norm in the space L^2. Define the space $H^s_{(V)} := D((-\Delta + V)^{\frac{s}{2}})$ endowed with the norm $\|\cdot\|_{s,V} = \|(\lambda_{j,V})^{\frac{s}{2}} \langle\cdot, e_{j,V}\rangle\|_{\ell^2}$. When D is the rectangle $(0,1)^d$ and $V(x_1, \dots, x_d) = V_1(x_1) + \dots + V_d(x_d)$, $V_k \in C^\infty([0,1], \mathbb{R})$, the eigenvalues and eigenfunctions of $-\Delta + V$ on D are of the form

$$\lambda_{j_1,\dots,j_d,V} = \lambda_{j_1,V_1} + \cdots + \lambda_{j_d,V_d}, \tag{1.4}$$

$$e_{j_1,\dots,j_d,V}(x_1,\dots,x_d) = e_{j_1,V_1}(x_1)\cdot\cdots\cdot e_{j_d,V_d}(x_d), \quad (x_1,\dots,x_d)\in D, \tag{1.5}$$

where $\{\lambda_{j,V_k}\}$ and $\{e_{j,V_k}\}$ are the eigenvalues and eigenfunctions of operator $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_k$ on $(0,1)$. Define the space

$$\mathcal{V} := \left\{ z \in L^2\colon \|z\|_{\mathcal{V}}^2 := \sum_{j_1,\dots,j_d=1}^{+\infty} \left| j_1^3 \cdot\cdots\cdot j_d^3 \langle z, e_{j_1,\dots,j_d,V}\rangle\right|^2 < +\infty \right\}. \tag{1.6}$$

Notice that, in the case $d=1$, the space $\mathcal{V}$ coincides with $H^3_{(V)}$. The eigenvalues and eigenfunctions of Dirichlet Laplacian on the interval $(0,1)$ are $\lambda_{k,0} = k^2\pi^2$ and $e_{k,0}(x) = \sqrt{2}\sin(k\pi x)$, $x\in(0,1)$. It is well known that for any $V\in L^2([0,1],\mathbb{R})$

$$\lambda_{k,V} = k^2\pi^2 + \int_0^1 V(x)\,\mathrm{d}x + r_k, \tag{1.7}$$

$$\|e_{k,V} - e_{k,0}\|_{L^\infty} \leqslant \frac{C}{k}, \tag{1.8}$$

$$\left\| \frac{\mathrm{d}e_{k,V}}{\mathrm{d}x} - \frac{\mathrm{d}e_{k,0}}{\mathrm{d}x} \right\|_{L^\infty} \leqslant C, \tag{1.9}$$

where $\sum_{k=1}^{+\infty} r_k^2 < +\infty$ (e.g., see [25]). For a Banach space X, we shall denote by $B_X(a,r)$ the open ball of radius $r>0$ centered at $a\in X$. For a set A, we write 2^A for the set consisting of all subsets of A. We denote by C a constant whose value may change from line to line.

2. Controllability of linearized system

2.1. Main result

In this section, we suppose that $d=1$ and $D=(0,1)$. For any $\tilde z \in H^3_{(V)}$, let $\mathcal{U}_t(\tilde z, 0)$ be the solution of (1.1)–(1.3) with $z_0 = \tilde z$ and $u=0$. Clearly,

$$\mathcal{U}_t(\tilde z, 0) = \sum_{j=1}^{+\infty} e^{-i\lambda_{j,V}t}\langle \tilde z, e_{j,V}\rangle e_{j,V}. \tag{2.1}$$

Lemma 2.1. *There is a sequence $T_n \to +\infty$ such that for any $\tilde z\in H^3_{(V)}$ we have $\mathcal{U}_{T_n}(\tilde z, 0)\to \tilde z$ in $H^3_{(V)}$.*

Proof. The proof uses the following well-known result (e.g., see [16]).

Lemma 2.2. *For any $\varepsilon>0$, $N\geqslant 1$ and $\alpha_j\in\mathbb{R}$, $j=1,\dots,N$, there is $k\in\mathbb{N}$ such that*

$$\sum_{j=1}^{N} \left| e^{i\alpha_j k} - 1\right| < \varepsilon.$$

Applying this lemma, we see that for any $\varepsilon>0$ and for sufficiently large $N\geqslant 1$, we have

$$\begin{aligned} \left\|\mathcal{U}_k(\tilde z,0) - \tilde z\right\|_{3,V}^2 &\leqslant \sum_{j\leqslant N} \left|e^{-i\lambda_{j,V}k} - 1\right|^2 \left|\lambda_{j,V}^{\frac32}\langle \tilde z, e_{j_1,\dots,j_d,V}\rangle\right|^2 \\ &\quad + 2\sum_{j>N}\left|\lambda_{j,V}^{\frac32}\langle \tilde z, e_{j_1,\dots,j_d,V}\rangle\right|^2 \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

for an appropriate choice of $k\in\mathbb{N}$. This proves Lemma 2.1. □

This subsection is devoted to the study of the linearization of (1.1), (1.2) around the trajectory $\mathcal{U}_t(\tilde z, 0)$:

$$i\dot z = -\frac{\partial^2 z}{\partial x^2} + V(x)z + u(t)Q(x)\mathcal{U}_t(\tilde z, 0), \tag{2.2}$$

$$z|_{\partial D} = 0, \tag{2.3}$$

$$z(0, x) = z_0. \tag{2.4}$$

Let S be the unit sphere in L^2. For $y \in S$, let T_y be the tangent space to S at $y \in S$:

$$T_y = \{z \in S\colon \operatorname{Re}\langle z, y\rangle = 0\}.$$

Lemma 2.3. *For any $z_0 \in T_{\tilde z} \cap H^2_{(0)}$ and $u \in L^1_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R})$, problem* (2.2)–(2.4) *has a unique solution $z \in C(\mathbb{R}_+, H^2_{(0)})$. Furthermore, if $R_t(\cdot,\cdot) : T_{\tilde z} \cap H^2_{(0)} \times L^1([0, t], \mathbb{R}) \to H^2_{(0)}$, $(z_0, u) \to z(t)$ is the resolving operator of the problem, then*

(i) *$R_t(z_0, u) \in T_{\mathcal{U}_t(\tilde z, 0)}$ for any $t \geqslant 0$,*
(ii) *The operator R_t is linear continuous from $T_{\tilde z} \cap H^2_{(0)} \times L^1([0, t], \mathbb{R})$ to $H^2_{(0)}$.*

Proof. The proof of existence and (ii) is standard (e.g., see [11]). To prove (i), notice that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Re}\langle R_t, \mathcal{U}_t\rangle &= \operatorname{Re}\langle \dot R_t, \mathcal{U}_t\rangle + \operatorname{Re}\langle R_t, \dot{\mathcal{U}}_t\rangle \\ &= \operatorname{Re}\left\langle i\left(\frac{\partial^2}{\partial x^2} - V\right)R_t - iu(t)Q(x)\mathcal{U}_t, \mathcal{U}_t\right\rangle + \operatorname{Re}\left\langle R_t, i\left(\frac{\partial^2}{\partial x^2} - V\right)\mathcal{U}_t\right\rangle \\ &= \operatorname{Re}\left\langle i\left(\frac{\partial^2}{\partial x^2} - V\right)R_t, \mathcal{U}_t\right\rangle + \operatorname{Re}\left\langle R_t, i\left(\frac{\partial^2}{\partial x^2} - V\right)\mathcal{U}_t\right\rangle = 0.\end{aligned}$$

Since $\operatorname{Re}\langle R_0, \mathcal{U}_0\rangle = \operatorname{Re}\langle z_0, \tilde z\rangle = 0$, we get (i). □

As (2.2)–(2.4) is a linear control problem, the controllability of system with $z_0 = 0$ is equivalent to that with any $z_0 \in T_{\tilde z}$. Henceforth, we take $z_0 = 0$ in (2.4). Let us rewrite this problem in the Duhamel form

$$z(t) = -i\int_0^t S(t-s)u(s)Q(x)\mathcal{U}_s(\tilde z, 0)\,\mathrm{d}s, \tag{2.5}$$

where $S(t) = e^{it(\frac{\partial^2}{\partial x^2} - V)}$ is the free evolution. Using (2.1) and (2.5), we obtain

$$\langle z(t), e_{m,V}\rangle = -i\sum_{k=1}^{+\infty} e^{-i\lambda_{m,V}t}\langle \tilde z, e_{k,V}\rangle Q_{mk}\int_0^t e^{i\omega_{mk}s}u(s)\,\mathrm{d}s, \quad m \geqslant 1, \tag{2.6}$$

where $\omega_{mk} = \lambda_m - \lambda_k$ and $Q_{mk} := \langle Qe_{m,V}, e_{k,V}\rangle$. Let $T_n \to +\infty$ be the sequence in Lemma 2.1. Then $e^{-i\lambda_{m,V}T_n} \to 1$ as $n \to +\infty$. Let us take $t = T_n$ in (2.6) and pass to the limit as $n \to +\infty$. For any $u \in L^1(\mathbb{R}_+, \mathbb{R})$ the right-hand side has a limit. Equality (2.6) implies that the following limit exists in the L^2-weak sense

$$R_\infty(0, u) := \lim_{n\to+\infty} z(T_n) = \lim_{n\to+\infty} R_{T_n}(0, u). \tag{2.7}$$

The choice of the sequence T_n implies that

$$\langle R_\infty(0, u), e_{m,V}\rangle = -i\sum_{k=1}^{+\infty}\langle \tilde z, e_{k,V}\rangle Q_{mk}\int_0^{+\infty} e^{i\omega_{mk}s}u(s)\,\mathrm{d}s. \tag{2.8}$$

Moreover, $R_\infty(0,u) \in T_{\tilde z}$. Indeed, using (2.7) and the convergence $\mathcal{U}_{T_n}(\tilde z, 0) \to \tilde z$ in $H^3_{(V)}$, we get

$$\operatorname{Re}\langle R_\infty(0,u), \tilde z\rangle = \lim_{n\to\infty} \operatorname{Re}\langle R_{T_n}(0,u), \mathcal{U}_{T_n}(\tilde z, 0)\rangle = 0,$$

by property (i).

For any $u \in L^1(\mathbb{R}_+, \mathbb{R})$, denote by $\check u$ the inverse Fourier transform of the function obtained by extending u as zero to $\mathbb{R}^*_-$:

$$\check u(\omega) := \int_0^{+\infty} e^{i\omega s} u(s)\,\mathrm{d}s. \tag{2.9}$$

Define the following spaces:

$$\begin{aligned}
\tilde\ell^2 &:= \Bigg\{ d = \{d_{mk}\}\colon\ \|d\|^2_{\tilde\ell^2} := |d_{11}|^2 + \sum_{m,k=1,\, m\neq k}^{+\infty} |d_{mk}|^2 < +\infty, \\
&\qquad d_{mm} = d_{11} \text{ and } d_{mk} = \bar d_{km} \text{ for all } m,k \geqslant 1 \Bigg\}, \\
\mathcal{B} &:= \Bigg\{ u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R})\colon\ \|u\|^2_{\mathcal{B}} := \sum_{p=1}^{+\infty} p^2 \|u\|^2_{L^2([p-1,p])} < +\infty \Bigg\}, \\
\mathcal{C} &:= \big\{ u \in L^1(\mathbb{R}_+, \mathbb{R})\colon\ \{\check u(\omega_{mk})\} \in \tilde\ell^2 \big\}.
\end{aligned}$$

The set of admissible controls is the Banach space

$$\Theta := u \in \mathcal{B} \cap \mathcal{C} \cap H^s(\mathbb{R}_+, \mathbb{R})$$

endowed with the norm $\|u\|_\Theta := \|u\|_{\mathcal{B}} + \|u\|_{L^1} + \|\{\check u(\omega_{mk})\}\|_{\tilde\ell^2} + \|u\|_{H^s}$, where $s \geqslant 1$ is any fixed constant. Clearly, the space Θ is nontrivial. The presence of the space $\mathcal{B}$ in the definition of Θ is motivated by the application to the nonlinear control system that we give in Section 3 (this guarantees that the trajectories of the nonlinear system with controls from $\mathcal{B}$ are bounded in the phase space). The space $\mathcal{C}$ in the definition of Θ ensures that the operator $R_\infty(0,\cdot)$ takes its values in $H^3_{(V)}$.

Lemma 2.4. *For any $\tilde z \in S \cap H^3_{(V)}$, $R_\infty(0,\cdot)$ is linear continuous mapping from Θ to $T_{\tilde z} \cap H^3_{(V)}$.*

Proof. Step 1. Let us admit that for any $m,k \geqslant 1$ we have

$$\left| \frac{m^3}{k^3} \langle Q e_{k,V}, e_{m,V}\rangle \right| \leqslant C. \tag{2.10}$$

Then (1.7), (2.8), (2.10) and the Schwarz inequality imply that

$$\begin{aligned}
\|R_\infty(0,u)\|^2_{3,V} &\leqslant C \sum_{m=1}^{+\infty} \big| m^3 \langle R_\infty(0,u), e_{m,V}\rangle \big|^2 \\
&\leqslant C \sum_{m=1}^{+\infty} \left| m^3 \langle \tilde z, e_{m,V}\rangle \langle Q e_{m,V}, e_{m,V}\rangle \int_0^{+\infty} u(s)\,\mathrm{d}s \right|^2 \\
&\quad + C\|\tilde z\|^2_{3,V} \sum_{m,k=1,\, m\neq k}^{+\infty} \left| \frac{m^3}{k^3} \langle Q e_{k,V}, e_{m,V}\rangle \int_0^{+\infty} e^{i\omega_{mk} s} u(s)\,\mathrm{d}s \right|^2 \\
&\leqslant C\|\tilde z\|^2_{3,V} \|u\|^2_\Theta < +\infty.
\end{aligned}$$

Step 2. Let us prove (2.10). Integration by parts gives

$$\langle Qe_{k,V}, e_{m,V}\rangle = \frac{1}{\lambda_{m,V}^2}\left\langle\left(-\frac{\partial^2}{\partial x^2}+V\right)(Qe_{k,V}), \left(-\frac{\partial^2}{\partial x^2}+V\right)(e_{m,V})\right\rangle$$
$$= \frac{1}{\lambda_{m,V}^2}\left(-2\frac{\partial Q}{\partial x}\frac{\partial e_{k,V}}{\partial x}\frac{\partial e_{m,V}}{\partial x}\Big|_{x=0}^{x=1} + \left\langle\frac{\partial}{\partial x}\left(-\frac{\partial^2}{\partial x^2}+V\right)(Qe_{k,V}), \frac{\partial e_{m,V}}{\partial x}\right\rangle\right.$$
$$\left. + \left\langle\left(-\frac{\partial^2}{\partial x^2}+V\right)(Qe_{k,V}), Ve_{m,V}\right\rangle\right).$$

In view of (1.4)–(1.9), this implies (2.10). □

We prove the controllability of (2.2), (2.3) under below condition with $d = 1$.

Condition 2.5. Suppose that D is the rectangle $(0, 1)^d$, $d \geqslant 1$, and the functions $V, Q \in C^\infty(\overline{D}, \mathbb{R})$ are such that

(i) $\inf_{p_1, j_1, \dots, p_d, j_d \geqslant 1} |(p_1 j_1 \cdot \dots \cdot p_d j_d)^3 Q_{pj}| > 0$, $Q_{pj} := \langle Qe_{p_1,\dots,p_d,V}, e_{j_1,\dots,j_d,V}\rangle$,
(ii) $\lambda_{i,V} - \lambda_{j,V} \neq \lambda_{p,V} - \lambda_{q,V}$ for all $i, j, p, q \geqslant 1$ such that $\{i, j\} \neq \{p, q\}$ and $i \neq j$.

See Appendix A for the proof of genericity of this condition. Let us introduce the set

$$\mathcal{E} := \{z \in S\colon \exists p, q \geqslant 1,\ p \neq q,\ z = c_p e_{p,V} + c_q e_{q,V},\ |c_p|^2\langle Qe_{p,V}, e_{p,V}\rangle - |c_q|^2\langle Qe_{q,V}, e_{q,V}\rangle = 0\}.$$

The following result is proved in the next subsection.

Theorem 2.6. *Under Condition* 2.5 *with $d = 1$, for any $\tilde{z} \in S \cap H^3_{(V)} \setminus \mathcal{E}$, the mapping $R_\infty(0, \cdot)\colon \Theta \to T_{\tilde{z}} \cap H^3_{(V)}$ admits a continuous right inverse, where the space $T_{\tilde{z}} \cap H^3_{(V)}$ is endowed with the norm of $H^3_{(V)}$. If $\tilde{z} \in S \cap H^3_{(V)} \cap \mathcal{E}$, then $R_\infty(0, \cdot)$ is not invertible.*

Remark 2.7. The invertibility of the mapping $R_T(0, \cdot)$ with finite $T > 0$ and $\tilde{z} = e_1$ is studied by Beauchard et al. [7]. They prove that for space dimension $d \geqslant 3$ the mapping is not invertible. By Beauchard [6], R_T is invertible in the case $d = 1$ and $\tilde{z} = e_1$. The case $d = 2$ is open to our knowledge.

Remark 2.8. Let us emphasize that the set $\{\omega_{mk}\}$ does not verify the gap condition (even in the case $d = 1$)

$$\inf_{(m,k)\neq(m',k')} |\omega_{mk} - \omega_{m'k'}| > 0.$$

Thus one cannot prove exact controllability in finite time near points, which are not eigenfunctions, using arguments based on the Ingham inequality.

2.2. Proof of Theorem 2.6

The proof of the theorem is based on the following proposition, which is proved in the next subsection.

Proposition 2.9. *If the sequence $\omega_m \in \mathbb{R}$, $m \geqslant 1$, is such that $\omega_1 = 0$ and $\sum_{m=2}^\infty \frac{1}{|\omega_m|^p} < +\infty$ for some $p \geqslant 1$ and $\omega_i \neq \omega_j$ for $i \neq j$, then there is a linear continuous operator A from ℓ_0^2 to Θ such that $\{\check{A}(d)(\omega_m)\} = d$ for any $d \in \ell_0^2$.*

The idea of the proof of Theorem 2.6 is to rewrite (2.8) in the form $d_{mk} = \check{u}(\omega_{mk})$ with $d = \{d_{mk}\} \in \tilde{\ell}^2$ and to apply the proposition. Notice that $\sum_{m,k=1, m\neq k}^\infty \frac{1}{\omega_{mk}^4} < +\infty$ and $\omega_{ij} \neq \omega_{pq}$ for all $i, j, p, q \geqslant 1$ such that $\{i, j\} \neq \{p, q\}$ and $i \neq j$. Let us take any $y \in T_{\tilde{z}} \cap H^3_{(V)}$. Define

$$d_{mk} := \frac{i\langle y, e_m\rangle\langle e_k, \tilde{z}\rangle - i\langle e_k, y\rangle\langle\tilde{z}, e_m\rangle}{Q_{mk}} + C_{mk},$$

where $C_{mk} \in \mathbb{C}$ and $e_k = e_{k,V}$. The fact that $\tilde{z} \in S$ implies

$$-i\sum_{k=1}^{+\infty}\langle\tilde{z}, e_k\rangle Q_{mk}d_{mk} = \sum_{k=1}^{+\infty}\langle y, e_m\rangle\left|\langle\tilde{z}, e_k\rangle\right|^2 - \sum_{k=1}^{+\infty}\langle e_k, y\rangle\langle\tilde{z}, e_m\rangle\langle\tilde{z}, e_k\rangle - i\sum_{k=1}^{+\infty}\langle\tilde{z}, e_k\rangle Q_{mk}C_{mk}$$
$$= \langle y, e_m\rangle - \langle\tilde{z}, e_m\rangle\langle\tilde{z}, y\rangle - i\sum_{k=1}^{+\infty}\langle\tilde{z}, e_k\rangle Q_{mk}C_{mk}.$$

By (2.8), we have $y = R_\infty(0, u)$, when

$$i\sum_{k=1}^{+\infty}\langle\tilde{z}, e_k\rangle Q_{mk}C_{mk} = -\langle\tilde{z}, e_m\rangle\langle\tilde{z}, y\rangle \tag{2.11}$$

for all $m \geqslant 1$. Thus if we show that there are $C_{mk} \in \mathbb{C}$ such that (2.11) is verified and $d = \{d_{mk}\} \in \tilde{\ell}^2$, then the proof of the theorem will be completed, in view of Proposition 2.9. Notice that, under Condition 2.5, we have

$$\sum_{m,k=1, m\neq k}^{+\infty}\left|\frac{\langle y, e_m\rangle\langle e_k, \tilde{z}\rangle}{Q_{mk}}\right|^2 \leqslant C\|y\|_{3,V}^2\|\tilde{z}\|_{3,V}^2 < +\infty.$$

Thus $\{d_{mk}\} \in \tilde{\ell}^2$, if $C_{mk} \in \mathbb{C}$ are such that

$$d_{mm} = \frac{i\langle y, e_m\rangle\langle e_m, \tilde{z}\rangle - i\langle e_m, y\rangle\langle\tilde{z}, e_m\rangle}{Q_{mm}} + C_{mm} = d_0, \tag{2.12}$$

$$C_{mk} = \overline{C}_{km}, \tag{2.13}$$

$$\sum_{m,k=1, m\neq k}^{+\infty}|C_{mk}|^2 < +\infty, \tag{2.14}$$

where $d_0 \in \mathbb{R}$. Let us show that, for an appropriate choice of d_0, there are C_{mk} satisfying (2.11)–(2.14). Since $y \in T_{\tilde{z}}$, we have $\langle\tilde{z}, y\rangle = i\,\mathrm{Im}\langle\tilde{z}, y\rangle$. We can rewrite (2.11) and (2.12) in the following form

$$\sum_{k=1}^{+\infty}\langle\tilde{z}, e_k\rangle Q_{mk}C_{mk} = -\langle\tilde{z}, e_m\rangle\,\mathrm{Im}\langle\tilde{z}, y\rangle, \tag{2.15}$$

$$d_{mm} = \frac{-2\,\mathrm{Im}(\langle y, e_m\rangle\langle e_m, \tilde{z}\rangle)}{Q_{mm}} + C_{mm} = d_0. \tag{2.16}$$

Case 1. Let as suppose that $\tilde{z} = ce_p$, where $c \in \mathbb{C}$, $|c| = 1$ and $p \geqslant 1$. Then (2.13)–(2.16) is verified for $C_{mk} = 0$, if $m \neq k$ and C_{mm} defined by (2.16) with $d_0 = \frac{\mathrm{Im}\langle\tilde{z}, y\rangle}{Q_{pp}}$.

Case 2. Suppose $\tilde{z} = c_pe_p + c_qe_q$, where $c_p, c_q \in \mathbb{C}$, $|c_p|^2 + |c_q|^2 = 1$ and $p \neq q$. For any $m \geqslant 1$, define C_{mm} by (2.16). If $m \neq p$, we set

$$C_{mp} := \frac{-c_m(\mathrm{Im}\langle\tilde{z}, y\rangle + Q_{mm}C_{mm})}{c_pQ_{mp}}, \tag{2.17}$$

where $c_m = 0$ for $m \neq q$, and $C_{mk} = 0$ for any $k \geqslant 1$ such that $k \neq m, p$. Then all the equations in (2.15) are verified, excepted the case $m = p$. Let us show that, for an appropriate choice of $d_0 \in \mathbb{R}$, this equation is also satisfied. Eq. (2.15) for $m = p$ is

$$c_pQ_{pp}C_{pp} + c_qQ_{pq}C_{pq} = -c_p\,\mathrm{Im}\langle\tilde{z}, y\rangle.$$

Using (2.17) for $m = q$ (taking $C_{pq} = \overline{C}_{qp}$) and (2.16) for $m = p$, we get

$$\begin{aligned}-c_p \operatorname{Im}\langle\tilde{z}, y\rangle &= c_p Q_{pp}\left(d_0 + \frac{2\operatorname{Im}(\langle y, e_p\rangle\langle e_p, \tilde{z}\rangle)}{Q_{pp}}\right) + c_q Q_{pq}\overline{\left(\frac{-c_q(\operatorname{Im}\langle\tilde{z}, y\rangle + Q_{qq}C_{qq})}{c_p Q_{qp}}\right)} \\ &= c_p Q_{pp}\left(d_0 + \frac{2\operatorname{Im}(\langle y, e_p\rangle\langle e_p, \tilde{z}\rangle)}{Q_{pp}}\right) + c_q Q_{pq}\left(\frac{-\bar{c}_q \operatorname{Im}\langle\tilde{z}, y\rangle}{\bar{c}_p Q_{qp}}\right) + c_q Q_{pq}\left(\frac{-\bar{c}_q Q_{qq}C_{qq}}{\bar{c}_p Q_{qp}}\right).\end{aligned}$$

Now using (2.16) for $m = q$, we rewrite this equality in an equivalent form

$$\left(|c_p|^2 Q_{pp} - |c_q|^2 Q_{qq}\right)d_0 = A$$

for some constant $A \in \mathbb{R}$. Thus if $\tilde{z}$ is such that $|c_p|^2 Q_{pp} - |c_q|^2 Q_{qq} \neq 0$, then we are able to find C_{mk} satisfying (2.13)–(2.16). If $|c_p|^2 Q_{pp} - |c_q|^2 Q_{qq} = 0$, then linear system (2.2), (2.3) is not controllable, since for any $u \in \Theta$ and $t \geqslant 0$ we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Im}\left\langle R_t(0,u), c_p e^{-i\lambda_p t}e_p - c_q e^{-i\lambda_q t}e_q\right\rangle \\ &\quad= \operatorname{Im}\left\langle i\left(\frac{\partial^2}{\partial x^2} - V\right)R_t(0,u) - iuQ\left(c_p e^{-i\lambda_p t}e_p + c_q e^{-i\lambda_q t}e_q\right), c_p e^{-i\lambda_p t}e_p - c_q e^{-i\lambda_q t}e_q\right\rangle \\ &\qquad+ \operatorname{Im}\left\langle R_t(0,u), i\left(\frac{\partial^2}{\partial x^2} - V\right)\left(c_p e^{-i\lambda_p t}e_p - c_q e^{-i\lambda_q t}e_q\right)\right\rangle \\ &\quad= \operatorname{Im}\left\langle -iuQ\left(c_p e^{-i\lambda_p t}e_p + c_q e^{-i\lambda_q t}e_q\right), c_p e^{-i\lambda_p t}e_p - c_q e^{-i\lambda_q t}e_q\right\rangle \\ &\quad= -u\left(|c_p|^2 Q_{pp} - |c_q|^2 Q_{qq}\right) = 0.\end{aligned}$$

This non-controllability property is a remark of Beauchard and Coron [8].

Case 3. Here we suppose that $\tilde{z} = \sum_{j=1}^{+\infty} c_j e_j$ with $c_p c_q c_r \neq 0$, and p, q, r are not equal to each other. If we define again C_{mp}, $m \neq p$, by (2.17) and $C_{mk} = 0$ for any $k \geqslant 1$ such that $k \neq m, p$, then the arguments of Case 2 give the following equation for d_0

$$\left(|c_p|^2 Q_{pp} - \sum_{m\neq p}|c_m|^2 Q_{mm}\right)d_0 = \tilde{A}$$

for some constant $\tilde{A} \in \mathbb{R}$. This implies that for any $\tilde{z}$ such that $|c_p|^2 Q_{pp} - \sum_{m\neq p}|c_m|^2 Q_{mm} \neq 0$, we can find C_{mk} satisfying (2.13)–(2.16). Let us suppose that

$$|c_p|^2 Q_{pp} - \sum_{m\neq p}|c_m|^2 Q_{mm} = 0. \tag{2.18}$$

In this case, we define C_{mp} by (2.17) only for integers $m \geqslant 1$ such that $m \neq p, q, r$ and $C_{mk} = 0$ for any $k \geqslant 1$ such that $k \neq m, p, q, r$. Then all the equations in (2.15) are verified, except for $m = p, q, r$. We take any $C_{qp} \in \mathbb{C}$ and choose C_{qr} and C_{rp} such that

$$c_p Q_{rp}C_{rp} + c_q Q_{rq}C_{rq} + c_p Q_{rr}C_{rr} = -c_r \operatorname{Im}\langle\tilde{z}, y\rangle, \tag{2.19}$$

$$c_p Q_{qp}C_{qp} + c_q Q_{qq}C_{qq} + c_r Q_{qr}C_{qr} = -c_q \operatorname{Im}\langle\tilde{z}, y\rangle. \tag{2.20}$$

Replacing the value of C_{qr} from (2.20) into (2.19), then the value of C_{pr} from (2.19) into (2.15) with $m = p$, and using (2.13), we get the following equation for d_0

$$\left(|c_p|^2 Q_{pp} + |c_q|^2 Q_{qq} - \sum_{m\neq p,q}|c_m|^2 Q_{mm}\right)d_0 = \tilde{\tilde{A}}$$

for some constant $\tilde{\tilde{A}} \in \mathbb{R}$. Equality (2.18) implies that

$$|c_p|^2 Q_{pp} + |c_q|^2 Q_{qq} - \sum_{m\neq p,q}|c_m|^2 Q_{mm} = 0$$

if and only if $|c_q|^2 Q_{qq} = 0$, which is not the case: $c_q \neq 0$, $Q_{qq} \neq 0$. Thus solution $d_0 \in \mathbb{R}$ exists, and the sequence C_{mk} is constructed for any $\tilde{z} \notin \mathcal{E}$.

2.3. Multidimensional case

In this section, we suppose that D is the rectangle $(0,1)^d$, $d \geqslant 1$, and $V(x_1,\dots,x_d) = V_1(x_1) + \dots + V_d(x_d)$, $V_k \in C^\infty([0,1],\mathbb{R})$. This subsection is devoted to the study of the linearization of (1.1), (1.2) around the trajectory $\mathcal{U}_t(\tilde{z},0)$:

$$i\dot{z} = -\Delta z + V(x)z + u(t)Q(x)\mathcal{U}_t(\tilde{z},0), \tag{2.21}$$

$$z|_{\partial D} = 0, \tag{2.22}$$

$$z(0,x) = z_0. \tag{2.23}$$

The proof of Theorem 2.6 does not work in the multidimensional case for a general $\tilde{z}$. Indeed, the well-known asymptotic formula for eigenvalues $\lambda_{k,V} \sim C_d k^{\frac{2}{d}}$ implies that the frequencies ω_{mk} are dense in $\mathbb{R}$ for space dimension $d \geqslant 3$. Thus the moment problem $\check{u}(\omega_{mk}) = d_{mk}$ cannot be solved in the space $L^1(\mathbb{R}_+,\mathbb{R})$ for a general $d_{mk} \in \tilde{\ell}^2$. The asymptotic formula for eigenvalues implies that the moment problem cannot be solved also in this case $d = 2$. Clearly, this does not imply the non-controllability of linearized system. Let us prove the controllability of (2.21), (2.22) for $\tilde{z} = e_{k,V}$. See our forthcoming publication for the case of a general $\tilde{z}$ and for an application to the nonlinear control problem.

For $\tilde{z} = e_{k,V}$ the mapping $R_\infty(0,u)$ is given by

$$\langle R_\infty(0,u), e_{m,V}\rangle = -iQ_{mk}\check{u}(\omega_{mk})$$

(cf. 2.8).

Lemma 2.10. *The mapping $R_\infty(0,\cdot)$ is linear continuous from Θ to $T_{e_{k,V}} \cap \mathcal{V}$, where $\mathcal{V}$ is defined by* (1.6).

Proof. Step 1. Let us admit that for any $m_j, k_j \geqslant 1$, $j = 1,\dots,d$ we have

$$\left|\frac{(m_1\cdot\dots\cdot m_d)^3}{(k_1\cdot\dots\cdot k_d)^3}\langle Qe_{k_1,\dots,k_d,V}, e_{m_1,\dots,m_d,V}\rangle\right| \leqslant C. \tag{2.24}$$

Then (2.8), (2.24) and the Schwarz inequality imply that

$$\begin{aligned}
\|R_\infty(0,u)\|_{\mathcal{V}}^2 &= \sum_{m_1,\dots,m_d=1}^{+\infty} \left|m_1^3\cdot\dots\cdot m_d^3\langle R_\infty(0,u), e_{m_1,\dots,m_d,V}\rangle\right|^2 \\
&\leqslant C\sum_{m=1}^{+\infty}\left|m_1^3\cdot\dots\cdot m_d^3\langle\tilde{z}, e_{m_1,\dots,m_d,V}\rangle\langle Qe_{m,V}, e_{m,V}\rangle\int_0^{+\infty} u(s)\,\mathrm{d}s\right|^2 \\
&\quad + C\|\tilde{z}\|_{\mathcal{V}}^2\sum_{m,k=1,\,m\neq k}^{+\infty}\left|\frac{(m_1\cdot\dots\cdot m_d)^3}{(k_1\cdot\dots\cdot k_d)^3}\langle Qe_{k_1,\dots,k_d,V}, e_{m_1,\dots,m_d,V}\rangle\int_0^{+\infty} e^{i\omega_{mk}s}u(s)\,\mathrm{d}s\right|^2 \\
&\leqslant C\|\tilde{z}\|_{\mathcal{V}}^2\|u\|_\Theta^2 < +\infty.
\end{aligned}$$

Step 2. Let us prove (2.24). To simplify notation, let us suppose that $d = 2$; the proof of the general case is similar. Let $V(x_1,x_2) = V_1(x_1) + V_2(x_2)$. Integration by parts gives

$$\begin{aligned}
\langle Qe_{k_1,k_2,V}, e_{m_1,m_2,V}\rangle &= \frac{1}{\lambda_{m_1,V_1}^2}\left\langle\left(-\frac{\partial^2}{\partial x_1^2} + V_1\right)(Qe_{k_1,k_2,V}), \left(-\frac{\partial^2}{\partial x_1^2} + V_1\right)(e_{m_1,m_2,V})\right\rangle \\
&= \frac{1}{\lambda_{m_1,V_1}^2}\left(\int_0^1 -2\frac{\partial Q}{\partial x_1}\frac{\partial e_{k_1,k_2,V}}{\partial x_1}\frac{\partial e_{m_1,m_2,V}}{\partial x_1}\bigg|_{x_1=0}^{x_1=1}\mathrm{d}x_2\right.
\end{aligned}$$

$$+\left\langle\frac{\partial}{\partial x_1}\left(-\frac{\partial^2}{\partial x_1^2}+V_1\right)(Qe_{k_1,k_2,V}),\,\frac{\partial e_{m_1,m_2,V}}{\partial x_1}\right\rangle$$
$$+\left\langle\left(-\frac{\partial^2}{\partial x_1^2}+V_1\right)(Qe_{k_1,k_2,V}),\,V_1e_{m_1,m_2,V}\right\rangle\Bigg)$$
$$=: I_1+I_2+I_3.$$

Again integrating by parts, we get

$$I_1=\frac{-2}{\lambda^2_{m_1,V_1}\lambda^2_{m_2,V_2}}\int_0^1\left(-\frac{\partial^2}{\partial x_2^2}+V_2\right)\left(\frac{\partial Q}{\partial x_1}\frac{\partial e_{k_1,k_2,V}}{\partial x_1}\right)\left(-\frac{\partial^2}{\partial x_2^2}+V_2\right)\frac{\partial e_{m_1,m_2,V}}{\partial x_1}\Bigg|_{x_1=0}^{x_1=1}\mathrm{d}x_2$$
$$=\frac{-2}{\lambda^2_{m_1,V_1}\lambda^2_{m_2,V_2}}\Bigg(-2\frac{\partial^2 Q}{\partial x_1\partial x_2}\frac{\partial^2 e_{k_1,k_2,V}}{\partial x_1\partial x_2}\frac{\partial^2 e_{m_1,m_2,V}}{\partial x_1\partial x_2}\Bigg|_{x_1=0}^{x_1=1}\Bigg|_{x_2=0}^{x_2=1}$$
$$+\int_0^1\frac{\partial}{\partial x_2}\left(-\frac{\partial^2}{\partial x_2^2}+V_2\right)\left(\frac{\partial Q}{\partial x_1}\frac{\partial e_{k_1,k_2,V}}{\partial x_1}\right)\frac{\partial^2 e_{m_1,m_2,V}}{\partial x_1\partial x_2}\Bigg|_{x_1=0}^{x_1=1}\mathrm{d}x_2$$
$$+\int_0^1\left(-\frac{\partial^2}{\partial x_2^2}+V_2\right)\left(\frac{\partial Q}{\partial x_1}\frac{\partial e_{k_1,k_2,V}}{\partial x_1}\right)V_2\frac{\partial e_{m_1,m_2,V}}{\partial x_1}\Bigg|_{x_1=0}^{x_1=1}\mathrm{d}x_2\Bigg).$$

In view of (1.4)–(1.9), this implies that

$$\left|\frac{(m_1\cdot\dots\cdot m_d)^3}{(k_1\cdot\dots\cdot k_d)^3}I_1\right|\leqslant C.$$

The terms I_2, I_3 are treated in the same way. We omit the details. □

We rewrite (2.8) in the form

$$\check{u}(\omega_{mk})=d_m,\tag{2.25}$$

where $d_m=\frac{\langle R_\infty(0,u),e_{m,V}\rangle}{-iQ_{mk}}$. We have $\sum_{m=1,\,m\neq k}^{\infty}\frac{1}{|\omega_{mk}|^d}<+\infty$ for fixed $k\geqslant 1$. Under Condition 2.5(i), $d_m\in\ell_0^2$. Applying Proposition 2.9, we obtain the following theorem:

Theorem 2.11. *Under Condition* 2.5*, the mapping* $R_\infty(0,\cdot):\Theta\to T_{e_{k,V}}\cap\mathcal{V}$ *admits a continuous right inverse, where the space* $T_{e_{k,V}}\cap\mathcal{V}$ *is endowed with the norm of* $\mathcal{V}$.

2.4. Proof of Proposition 2.9

The construction of the operator A is based on the following lemma.

Lemma 2.12. *Under the conditions of Proposition* 2.9*, for any* $d\in\ell_0^2$ *and* $\varepsilon>0$*, there is* $u\in B_\Theta(0,\varepsilon)$ *such that* $\{\check{u}(\omega_m)\}=d$.

Proof of Proposition 2.9. Let d^n be any orthonormal basis in ℓ_0^2. Applying Lemma 2.12, we find a sequence $u_n\in B_\Theta(0,\frac{1}{n})$ such that $\{\check{u}_n(\omega_m)\}=d^n$. For any $d\in\ell_0^2$, there is $c\in\ell^2$ such that $d=\sum_{n=1}^{+\infty}c_nd^n$. Let us define A in the following way

$$A(d)=\sum_{n=1}^{+\infty}c_nu_n.$$

As $u_n\in B_\Theta(0,\frac{1}{n})$, this sum converges in Θ:

$$\left\|A(d)\right\|_\Theta \leqslant \sum_{n=1}^{+\infty}|c_n|\|u_n\|_\Theta \leqslant \left(\sum_{n=1}^{+\infty}|c_n|^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{+\infty}\|u_n\|_\Theta^2\right)^{\frac{1}{2}} \leqslant C\|d\|_{\ell_0^2}.$$

Thus $A:\ell_0^2\to\Theta$ is linear continuous and $\{\check{A}(d)(\omega_m)\}=d$, by construction. □

Proof of Lemma 2.12. Let us take any $d\in\ell_0^2$ and $\varepsilon>0$ and introduce the functional

$$\mathcal{H}(u):=\left\|\left\{\check{u}(\omega_m)\right\}-d\right\|_{\ell_0^2}^2=\sum_{m=1}^{+\infty}\left|\check{u}(\omega_m)-d_m\right|^2$$

defined on the space Θ.

Step 1. First, let us show that there is $u_0\in\overline{B_\Theta(0,\varepsilon)}$ such that

$$\mathcal{H}(u_0)=\inf_{u\in\overline{B_\Theta(0,\varepsilon)}}\mathcal{H}(u). \tag{2.26}$$

To this end, let $u_n\in\overline{B_\Theta(0,\varepsilon)}$ be an arbitrary minimizing sequence. Since $\mathcal{B}\cap H^s(\mathbb{R}_+,\mathbb{R})$ is reflexive, without loss of generality, we can assume that there is $u_0\in\overline{B_{\mathcal{B}\cap H^s(\mathbb{R}_+,\mathbb{R})}(0,\varepsilon)}$ such that $u_n\rightharpoonup u_0$ in $\mathcal{B}\cap H^s(\mathbb{R}_+,\mathbb{R})$. Using the compactness of the injection $H^s([0,N])\to C([0,N])$ for any $N>0$ and a diagonal extraction, we can assume that $u_n(t)\to u_0(t)$ uniformly for $t\in[0,N]$. The Fatou lemma implies that

$$\int_0^{+\infty}\left|u_0(s)\right|\mathrm{d}s\leqslant\liminf_{n\to\infty}\int_0^{+\infty}\left|u_n(s)\right|\mathrm{d}s\leqslant\varepsilon.$$

Again extracting a subsequence, if it is necessary, one gets $\{\check{u}_n(\omega_m)\}\rightharpoonup\{\check{u}_0(\omega_m)\}$ in ℓ_0^2 as $n\to+\infty$. Indeed, the tails on $[T,+\infty)$, $T\gg1$, of the integrals (2.9) are small uniformly in n (this comes from the boundedness of u_n in $\mathcal{B}$), and on the finite interval $[0,T]$ the convergence is uniform.

This implies that $u_0\in\Theta$ and

$$\mathcal{H}(u_0)\leqslant\inf_{u\in\overline{B_\Theta(0,\varepsilon)}}\mathcal{H}(u).$$

The fact that $u_0\in\overline{B_\Theta(0,\varepsilon)}$ follows from the Fatou lemma and lower weak semicontinuity of norms. Thus we have (2.26).

Step 2. To complete the proof, we need to show that $\mathcal{H}(u_0)=0$. Suppose, by contradiction, that $\mathcal{H}(u_0)>0$. As we shall see below, this implies that there is $v\in\overline{B_\Theta(0,\varepsilon)}$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}\big((1-t)u_0+tv\big)\bigg|_{t=0}<0. \tag{2.27}$$

Since $(1-t)u_0+tv\in\overline{B_\Theta(0,\varepsilon)}$ for all $t\in[0,1]$, (2.27) is a contradiction to (2.26).

To construct such a function v, notice that the derivative is given explicitly by

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}\big((1-t)u_0+tv\big)\bigg|_{t=0}=2\sum_{m=1}^{+\infty}\mathrm{Re}\Big[\big(\check{v}(\omega_m)-\check{u}_0(\omega_m)\big)\overline{\big(\check{u}_0(\omega_m)-d_m\big)}\Big].$$

In view of this equality, the existence of v follows immediately from the following lemma:

Lemma 2.13. *Under the conditions of Proposition* 2.9, *the set*

$$U:=\left\{\left\{\check{u}(\omega_m)\right\}:\ u\in B_\Theta(0,\varepsilon)\right\}$$

is dense in ℓ_0^2.

Proof. Suppose that $h \in \ell_0^2$ is orthogonal to U. Then for any $u \in B_\Theta(0,\varepsilon) \cap C_0^\infty((0,+\infty))$ we have

$$\sum_{m=1}^{+\infty} \check{u}(\omega_m)\overline{h_m} = 0. \tag{2.28}$$

Replacing in this equality $\check{u}(\omega_m)$ by its integral representation, we get integrating by parts

$$\begin{aligned} 0 &= \sum_{m=1}^{+\infty} \int_0^{+\infty} e^{i\omega_m s} u(s)\, \mathrm{d}s\, \overline{h_m} = \int_0^{+\infty} P_p(s) u^{(p)}(s)\, \mathrm{d}s\, \overline{h_1} + \sum_{m=2}^{+\infty} \int_0^{+\infty} \frac{e^{i\omega_m s}}{(-i\omega_m)^p} u^{(p)}(s)\, \mathrm{d}s\, \overline{h_m} \\ &= \int_0^{+\infty} u^{(p)}(s) \left(P_p(s)\overline{h_1} + \sum_{m=2}^{+\infty} \frac{e^{i\omega_m s}}{(-i\omega_m)^p} \overline{h_m} \right) \mathrm{d}s = 0, \end{aligned}$$

where P_p is a polynomial of degree $p \geqslant 1$. Since this equality holds for any $u \in B_\Theta(0,\varepsilon) \cap C_0^\infty((0,+\infty))$, there is a polynomial $\tilde{P}_{p-1}(s)$ of degree $p-1$ such that for any $s \geqslant 0$

$$P_p(s)\overline{h_1} + \sum_{m=2}^{+\infty} \frac{e^{i\omega_m s}}{(-i\omega_m)^p} \overline{h_m} = \tilde{P}_{p-1}(s).$$

By Lemma 2.14, we have $h_m = 0$ for any $m \geqslant 2$. Equality (2.28) implies that $h_1 = 0$. This proves that U is dense. □

The following lemma is a generalization of Lemma 3.10 in [22].

Lemma 2.14. *Suppose that $r_j \in \mathbb{R}^*$ and $r_k \neq r_j$ for $k \neq j$ and P_p is a polynomial of degree $p \geqslant 1$. If*

$$\sum_{j=1}^{\infty} c_j e^{ir_j s} = P_p(s) \tag{2.29}$$

for any $s \geqslant 0$ and for some sequence $c_j \in \mathbb{C}$ such that $\sum_{j=1}^{\infty} |c_j| < \infty$, then $c_j = 0$ for all $j \geqslant 1$ and $P_p \equiv 0$.

Proof. Since the sum in the left-hand side of (2.29) is bounded in s, the polynomial $P_p(s)$ is constant. The case of constant right-hand side follows from Lemma 3.10 in [22]. □

3. Controllability of nonlinear system

3.1. Well-posedness of Schrödinger equation

In this section, we suppose that $d = 1$, $D = (0,1)$. We consider the following Schrödinger equation

$$i\dot{z} = -\frac{\partial^2 z}{\partial x^2} + V(x)z + u(t)Q(x)z + v(t)Q(x)y, \tag{3.1}$$

$$z|_{\partial D} = 0, \tag{3.2}$$

$$z(0,x) = z_0(x). \tag{3.3}$$

See Proposition 2 in [10] for the proof of well-posedness of this system with $V = 0$. Here we prove well-posedness in the case of $V \neq 0$ and we give an estimate for the solution which is important for the study of the controllability property.

Proposition 3.1. *For any $z_0 \in H^3_{(V)}$, $u, v \in L^1(\mathbb{R}_+, \mathbb{R}) \cap \mathcal{B}$ and $y \in C(\mathbb{R}_+, H^3_{(V)})$, problem* (3.1)–(3.3) *has a unique solution $z \in C(\mathbb{R}_+, H^3_{(V)})$. Furthermore, there is a constant $C > 0$ such that*

$$\sup_{t\in\mathbb{R}_+}\big\|z(t)\big\|_{3,V}\leqslant C\Big(\|z_0\|_{3,V}+\sup_{t\in\mathbb{R}_+}\big\|y(t)\big\|_{3,V}\big(\|v\|_{L^1(\mathbb{R}_+)}+\|v\|_{\mathcal{B}}\big)\Big)\times\exp\big(C\big(\|u\|_{L^1(\mathbb{R}_+)}+1\big)\exp\big(\|u\|_{\mathcal{B}}^2\big)\big). \tag{3.4}$$

If $v=0$, then for all $t\geqslant 0$ we have

$$\big\|z(t)\big\|=\|z_0\|. \tag{3.5}$$

Proof. The proof follows the ideas of Proposition 2 in [10]. We give all the details for the sake of completeness.

Let us rewrite (3.1)–(3.3) in the Duhamel form

$$z(t)=S(t)z_0-i\int_0^t S(t-s)\big[u(s)Qz(s)+v(s)Qy(s)\big]\,\mathrm{d}s. \tag{3.6}$$

For any $u\in L^1(\mathbb{R}_+,\mathbb{R})\cap\mathcal{B}$ and $z\in C(\mathbb{R}_+,H^3_{(V)})$, we estimate the function

$$G_t(z):=\int_0^t S(-s)\big(u(s)Qz(s)\big)\,\mathrm{d}s.$$

Integration by parts gives (we write λ_j, e_j instead of $\lambda_{j,V}, e_{j,V}$)

$$\begin{aligned}\big\langle Qz(s),e_j\big\rangle&=\frac{1}{\lambda_j}\Big\langle\Big(-\frac{\partial^2}{\partial x^2}+V\Big)(Qz),e_j\Big\rangle\\&=\frac{1}{\lambda_j^2}\Big\langle\Big(-\frac{\partial^2}{\partial x^2}+V\Big)(Qz),\Big(-\frac{\partial^2}{\partial x^2}+V\Big)e_j\Big\rangle\\&=\frac{1}{\lambda_j^2}\frac{\partial^2}{\partial x^2}(Qz)\frac{\partial}{\partial x}e_j\Big|_{x=0}^{x=1}+\frac{1}{\lambda_j^2}\Big(\Big\langle V\Big(-\frac{\partial^2}{\partial x^2}+V\Big)(Qz),e_j\Big\rangle+\Big\langle\frac{\partial}{\partial x}\Big(-\frac{\partial^2}{\partial x^2}+V\Big)(Qz),\frac{\partial}{\partial x}e_j\Big\rangle\Big)\\&=:I_j+J_j.\end{aligned}$$

Thus

$$\big\|G_t(z)\big\|_{3,V}^2=\sum_{j=1}^{+\infty}\Big(j^3\int_0^t e^{i\lambda_j s}u(s)\big\langle Qz(s),e_j\big\rangle\mathrm{d}s\Big)^2=\sum_{j=1}^{+\infty}\Big(j^3\int_0^t e^{i\lambda_j s}u(s)(I_j+J_j)\,\mathrm{d}s\Big)^2. \tag{3.7}$$

Using (1.9), we get

$$\Big\langle\frac{\partial}{\partial x}\Big(-\frac{\partial^2}{\partial x^2}+V\Big)Qz,\frac{\partial}{\partial x}e_j\Big\rangle=j\pi\Big\langle\frac{\partial}{\partial x}\Big(-\frac{\partial^2}{\partial x^2}+V\Big)Qz,\sqrt{2}\cos(j\pi x)\Big\rangle+s_j(z),$$

where $|s_j(z)|\leqslant C\|z\|_{3,V}$ for all $j\geqslant 1$. The definition of J_j, the fact that $\{\sqrt{2}\cos(j\pi x)\}$ is an orthonormal system in L^2, (1.7) and the Minkowski inequality yield

$$\sum_{j=1}^{+\infty}\Big(j^3\int_0^t e^{i\lambda_j s}u(s)J_j\,\mathrm{d}s\Big)^2\leqslant C\Big(\int_0^t\big|u(s)\big|\big\|z(s)\big\|_{3,V}\,\mathrm{d}s\Big)^2. \tag{3.8}$$

On the other hand, (1.9) implies that

$$\frac{\partial^2}{\partial x^2}(Qz)\frac{\partial}{\partial x}e_j\Big|_{x=0}^{x=1}=j\pi\frac{\partial^2}{\partial x^2}(Qz)\sqrt{2}\cos(j\pi x)\Big|_{x=0}^{x=1}+\tilde{s}_j(z)=:jc_j(z)+\tilde{s}_j(z),$$

where $|\tilde{s}_j|\leqslant C\|z\|_{3,V}$ for all $j\geqslant 1$. Again applying the Minkowski inequality, we obtain

$$\sum_{j=1}^{+\infty}\left(\frac{j^3}{\lambda_j^2}\int_0^t e^{i\lambda_j s}u(s)\tilde{s}_j(z)\,\mathrm{d}s\right)^2 \leqslant C\left(\int_0^t \left|u(s)\right|\left\|z(s)\right\|_{3,V}\,\mathrm{d}s\right)^2. \tag{3.9}$$

Since $c_j(z)$ depends on the parity of j, without loss of generality, we can assume that $c(z) := c_j(z)$ does not depend on j. Thus we cannot conclude as in the case of J_j. Here we use the fact that $u \in \mathcal{B}$. Let $P \geqslant 1$ be an integer such that $P \leqslant t < P+1$. Using the Cauchy–Schwarz and the Ingham inequalities, we obtain

$$\begin{aligned}
&\sum_{j=1}^{+\infty}\left(\int_0^t e^{i\lambda_j s}u(s)c(z)\,\mathrm{d}s\right)^2 \\
&\quad= \sum_{j=1}^{+\infty}\left(\left(\int_P^t + \sum_{p=1}^{P}\int_{p-1}^{p}\right)e^{i\lambda_j s}u(s)c(z)\,\mathrm{d}s\right)^2 \\
&\quad\leqslant 2\sum_{j=1}^{+\infty}\left(\int_P^t e^{i\lambda_j s}u(s)c(z)\,\mathrm{d}s\right)^2 + 2\sum_{j=1}^{+\infty}\left(\sum_{p=1}^{P}\frac{1}{p^2}\right)\left(\sum_{p=1}^{P}p^2\left(\int_{p-1}^{p} e^{i\lambda_j s}u(s)c(z)\,\mathrm{d}s\right)^2\right) \\
&\quad\leqslant C\left\|u(s)c(z)\right\|^2_{L^2([P,t])} + C\sum_{p=1}^{P}p^2\sum_{j=1}^{+\infty}\left(\int_{p-1}^{p} e^{i\lambda_j s}u(s)c(z)\,\mathrm{d}s\right)^2 \\
&\quad\leqslant C\left\|u(s)c(z)\right\|^2_{L^2([P,t])} + C\sum_{p=1}^{P}p^2\left\|u(s)c(z)\right\|^2_{L^2([p-1,p])} \\
&\quad\leqslant C\int_0^t w(s)\left\|z(s)\right\|^2_{3,V}\,\mathrm{d}s,
\end{aligned}$$

where $w(s) = |u(s)|^2\chi_{[P,t]}(s) + \sum_{p=1}^{P}p^2|u(s)|^2\chi_{[p-1,p]}(s)$. Notice that

$$\int_0^t w(s)\,\mathrm{d}s \leqslant \|u\|^2_{\mathcal{B}} \quad \text{for all } t \geqslant 0. \tag{3.10}$$

Combining (3.7)–(3.10), we get

$$\left\|G_t(z)\right\|_{3,V} \leqslant C\left(\int_0^t w(s)\left\|z(s)\right\|^2_{3,V}\,\mathrm{d}s\right)^{\frac{1}{2}} + C\int_0^t \left|u(s)\right|\left\|z(s)\right\|_{3,V}\,\mathrm{d}s. \tag{3.11}$$

The quantity

$$\tilde{G}_t(f) := \int_0^t S(-s)\big(v(s)Qy(s)\big)\,\mathrm{d}s$$

is estimated in a similar way

$$\begin{aligned}
\|\tilde{G}_t\|_{3,V} &\leqslant C\left(\int_0^t \tilde{w}(s)\left\|y(s)\right\|^2_{3,V}\,\mathrm{d}s\right)^{\frac{1}{2}} + C\int_0^t \left|v(s)\right|\left\|y(s)\right\|_{3,V}\,\mathrm{d}s \\
&\leqslant C\sup_{s\in[0,T]}\left\|y(s)\right\|_{3,V}\big(\|v\|_{L^1(\mathbb{R}_+)} + \|v\|_{\mathcal{B}}\big),
\end{aligned} \tag{3.12}$$

where $\tilde{w}(s) = |v(s)|^2\chi_{[P,t]}(s) + \sum_{p=1}^{P}p^2|v(s)|^2\chi_{[p-1,p]}(s)$.

Existence of a solution is obtained easily from (3.11) and (3.12), by a fixed point theorem (cf. Proposition 2 in [10]). Uniqueness follows from (3.4).

Let us prove (3.4). From (3.6) and (3.11) we have

$$\begin{aligned}\|z(t)\|_{3,V}^2 &\leqslant C\big(\|z_0\|_{3,V}^2 + \|\tilde{G}_t\|_{3,V}^2 + \|G_t\|_{3,V}^2\big)\\ &\leqslant C\left(\|z_0\|_{3,V}^2 + \|\tilde{G}_t\|_{3,V}^2 + \int_0^t w(s)\|z(s)\|_{3,V}^2\,\mathrm{d}s + \left(\int_0^t |u(s)|\|z(s)\|_{3,V}\,\mathrm{d}s\right)^2\right).\end{aligned}$$

The Gronwall inequality implies

$$\|z(t)\|_{3,V}^2 \leqslant C\left(\|z_0\|_{3,V}^2 + \|\tilde{G}_t\|_{3,V}^2 + \left(\int_0^t |u(s)|\|z(s)\|_{3,V}\,\mathrm{d}s\right)^2\right)\exp\left(C\int_0^t w(s)\,\mathrm{d}s\right).$$

Taking the square root of this inequality, using (3.10) and the Gronwall inequality, we obtain

$$\begin{aligned}\|z(t)\|_{3,V} &\leqslant C\big(\|z_0\|_{3,V} + \|\tilde{G}_t\|_{3,V}\big)\exp\left(C\left(\int_0^t w(s)\,\mathrm{d}s + \int_0^t |u(s)|\,\mathrm{d}s\exp\left(\int_0^t w(s)\,\mathrm{d}s\right)\right)\right)\\ &\leqslant C\big(\|z_0\|_{3,V} + \|\tilde{G}_t\|_{3,V}\big)\exp\big(C\big(\|u\|_{L^1(\mathbb{R}_+)} + 1\big)\exp\big(\|u\|_{\mathcal{B}}^2\big)\big).\end{aligned}$$

In view of (3.12), this completes the proof of the proposition. □

Remark 3.2. Let us notice that, one should not expect to have a well-posedness property in any Sobolev space H^k with controls in L^1. Indeed, exact controllability property in H^3, proved by Beauchard and Laurent [10] in the case $d = 1$, implies that the problem is not well posed in spaces $H^{3+\sigma}$ for any $\sigma > 0$ (a point $z_1 \in H^3 \setminus H^{3+\sigma}$ would not be accessible from a point $z_0 \in H^{3+\sigma}$). Schrödinger equation is well-posed in higher Sobolev spaces, when control u is more regular.

Corollary 3.3. *Denote by $\mathcal{U}_t(\cdot,\cdot): H_{(V)}^3 \times L^1(\mathbb{R}_+,\mathbb{R}) \cap \mathcal{B} \to H_{(V)}^3$ the resolving operator of* (1.1), (1.2). *Then $\mathcal{U}_t(\cdot,\cdot)$ is locally Lipschitz continuous, i.e., for any $\delta > 0$ there is $C > 0$ such that*

$$\sup_{t\in\mathbb{R}_+}\big\|\mathcal{U}_t(z_0,u) - \mathcal{U}_t(z_0',u')\big\|_{3,V} \leqslant C\big\|(z_0,u) - (z_0',u')\big\|_{H_{(V)}^3\times L^1(\mathbb{R}_+,\mathbb{R})\cap\mathcal{B}} \tag{3.13}$$

for all $(z_0,u), (z_0',u') \in B_{H_{(V)}^3\times L^1(\mathbb{R}_+,\mathbb{R})\cap\mathcal{B}}(0,\delta)$, where $L^1(\mathbb{R}_+,\mathbb{R}) \cap \mathcal{B}$ is endowed with the norm $\|\cdot\|_{L^1(\mathbb{R}_+,\mathbb{R})\cap\mathcal{B}} := \|\cdot\|_{L^1} + \|\cdot\|_{\mathcal{B}}$.

Proof. Notice that $z(t) := \mathcal{U}_t(z_0,u) - \mathcal{U}_t(z_0',u')$ is a solution of problem

$$\begin{aligned} i\dot{z} &= -\frac{\partial^2 z}{\partial^2 x} + u(t)Q(x)z + \big(u(t) - u'(t)\big)Q(x)\mathcal{U}_t\big(z_0',u'\big),\\ z|_{\partial D} &= 0,\\ z(0,x) &= z_0(x) - z_0'(x).\end{aligned}$$

Applying Proposition 3.1, we get (3.13). □

3.2. Exact controllability in infinite time

For any control $u \in \Theta$, problem (3.1), (3.2) is well-posed in Sobolev space $H_{(V)}^3$. Equality (3.5) implies that it suffices to consider the controllability properties of (3.1), (3.2) on the unit sphere S in L^2. Let $\mathcal{U}_\infty(z_0,u)$ be the $H_{(V)}^3$-weak ω-limit set of the trajectory corresponding to control $u \in \Theta$ and initial condition $z_0 \in H_{(V)}^3$:

$$\mathcal{U}_\infty(z_0,u) := \big\{z \in H_{(V)}^3 : \mathcal{U}_{t_n}(z_0,u) \rightharpoonup z \text{ in } H_{(V)}^3 \text{ for some } t_n \to +\infty\big\}. \tag{3.14}$$

By (3.4), $\mathcal{U}_t(z_0,u)$ is bounded in $H_{(V)}^3$, thus $\mathcal{U}_\infty(z_0,u)$ is non-empty.

Definition 3.4. We say that (3.1), (3.2) is exactly controllable in infinite time in subset $H \subset S$, if for any $z_0, z_1 \in H$ there is a control $u \in \Theta$ such that $z_1 \in \mathcal{U}_\infty(z_0, u)$.

Below theorem is one of the main results of this paper.

Theorem 3.5. *Under Condition* 2.5, *for any* $\tilde{z} \in S \cap H^3_{(V)}$ *there is* $\delta > 0$ *such that problem* (3.1), (3.2) *is exactly controllable in infinite time in* $S \cap B_{H^3_{(V)}}(\tilde{z}, \delta)$.

See Section 3.3 for the proof.

Remark 3.6. Let us emphasize that the novelty of Theorem 3.5 with respect to the previous result proved for (3.1), (3.2) in [23] (see Theorem 3.1) is that the controllability here is realized with controls which have small norms.

Working in higher Sobolev spaces, one can prove similar exact controllability results with more regular controls. For example:

Theorem 3.7. *Under Condition* 2.5, *for any* $\tilde{z} \in S \cap H^{3+\sigma}_{(V)}$, $\sigma \in (0, 2]$ *there is* $\delta > 0$ *such that problem* (3.1), (3.2) *is exactly controllable in infinite time in* $S \cap B_{H^{3+\sigma}_{(V)}}(\tilde{z}, \delta)$ *with controls* $u \in W^{1,1}(\mathbb{R}_+, \mathbb{R}) \cap H^s(\mathbb{R}_+, \mathbb{R})$ *for any* $s \geqslant 1$.

These local exact controllability properties imply the following global exact controllability result.

Theorem 3.8. *Under Condition* 2.5, *problem* (3.1), (3.2) *is exactly controllable in infinite time in* $S \cap H^3_{(V)}$ *in the following sense*: *for any* $z_0 \in S \cap H^{3+\sigma}_{(V)}$, $\sigma \in (0, 2]$, *and* $z_1 \in S \cap H^3_{(V)}$ *there is a control* $u \in L^1(\mathbb{R}_+, \mathbb{R})$ *such that* $z_1 \in \mathcal{U}_\infty(z_0, u)$.

Proof. Let $\gamma : [0, 1] \to S \cap H^3_{(V)}$ be any continuous function such that $\gamma(0) = z_0$, $\gamma(1) = z_1$ and $\gamma(s) \in H^{3+\sigma}_{(V)}$ for any $s \in [0, 1)$. Using the compactness of the curve γ and Theorem 3.7, we prove that there is a control v and time $T > 0$ such that $\mathcal{U}_T(z_0, v) \in B_{H^3_{(V)}}(z_1, \delta_{z_1})$, where $\delta_{z_1} > 0$ is the constant in Theorem 3.5 corresponding to z_1. This completes the proof. □

Remark 3.9. We do not know if problem (1.1)–(1.3) is well posed in the space $\mathcal{V}$ for $d \geqslant 2$ with Θ-controls. Well-posedness in $\mathcal{V}$ with $u \in \Theta$ would imply the controllability of the multidimensional problem. The nonlinear problem's solution is in $\mathcal{V}$ for more regular controls.

3.3. Proof of Theorem 3.5

The proof is based on an inverse mapping theorem for multivalued functions. We apply the inverse mapping theorem established by Nachi and Penot [21], which suits well to the setting of Schrödinger equation. For the reader's convenience, we recall the statement of their result in Appendix A (see Theorem A.3).

Let us first slightly modify the definition (3.14) of the set $\mathcal{U}_\infty(z_0, u)$. Let $T_n \to +\infty$ be the sequence defined in Section 2.1. Define

$$\mathcal{U}_\infty(z_0, u) := \left\{z \in H^3_{(V)} : \mathcal{U}_{T_{n_k}}(z_0, u) \rightharpoonup z \text{ in } H^3 \text{ for some } n_k \to +\infty\right\}. \tag{3.15}$$

Consider the multivalued function

$$\begin{aligned} \mathcal{U}_\infty(\cdot,\cdot) : S \cap H^3_{(V)} \times \Theta &\to 2^{S \cap H^3_{(V)}}, \\ (z_0, u) &\to \mathcal{U}_\infty(z_0, u). \end{aligned}$$

Since the result of Nachi and Penot is stated in the case of Banach spaces, we cannot apply it directly to $\mathcal{U}_\infty$. Following Beauchard and Laurent [10], we project the system onto the tangent space $T_{\tilde{z}}$. We apply Theorem A.3 to the following multivalued function

$$\tilde{\mathcal{U}}_\infty(\cdot,\cdot): T_{\tilde z} \cap H^3_{(V)} \times \Theta \to 2^{T_{\tilde z}\cap H^3_{(V)}},$$
$$(z_0,u) \to P\mathcal{U}_\infty\big(P^{-1}z_0,u\big),$$

where P is the orthogonal projection in L^2 onto $T_{\tilde z}$, i.e., $Pz = z - \mathrm{Re}\langle z,\tilde z\rangle \tilde z$, $z \in L^2$. Notice that $P^{-1}: B_{T_{\tilde z}}(0,\delta) \to S$ is well defined for sufficiently small $\delta > 0$. By the definition of T_n, we have $\lim_{n\to+\infty}\mathcal{U}_{T_n}(\tilde z,0) = \tilde z$. Hence (3.15) implies that $\mathcal{U}_\infty(\tilde z,0) = \tilde z$ and $\tilde{\mathcal{U}}_\infty(0,0) = \{0\}$. If we show that $\tilde{\mathcal{U}}_\infty$ is strictly differentiable at (x_0,y_0) with $x_0 = (0,0) \in T_{\tilde z} \cap H^3_{(V)} \times \Theta$ and $y_0 = 0 \in T_{\tilde z} \cap H^3_{(V)}$ (see Definition A.2), and the derivative admits a right inverse, then Theorem 3.5 will be proved as a consequence of Theorem A.3.

Proposition 3.10. *The multifunction $\tilde{\mathcal{U}}_\infty$ is strictly differentiable at $(0,0) \in T_{\tilde z} \cap H^3_{(V)} \times \Theta$ in the sense of Definition* A.2. *Moreover, the differential is the mapping*

$$R_\infty(\cdot,\cdot): T_{\tilde z} \cap H^3_{(V)} \times \Theta \to T_{\tilde z} \cap H^3_{(V)},$$
$$(z_0,u) \to R_\infty(z_0,u),$$

where R_∞ is defined in Section 2.1.

Proof of Theorem 3.5. Case 1. Let us suppose that $\tilde z \in S \cap H^3_{(V)} \setminus \mathcal{E}$. For any $(z_0,u) \in B_{T_{\tilde z}\cap H^3_{(V)}\times\Theta}(0,\delta)$, the set $\tilde{\mathcal{U}}_\infty(z_0,u)$ is closed and non-empty, if $\delta > 0$ is sufficiently small. The mapping R_∞ is invertible in view of Theorem 2.6. Thus Theorem A.3 completes the proof.

Remark 3.11. Let us point out that in Case 1 the controls u can be chosen such that $u(0) = \cdots = u^{(s-1)}(0) = 0$.

Case 2. In the case $\tilde z \in S \cap H^3_{(V)} \cap \mathcal{E}$, the linearized system (2.2), (2.3) is not controllable, and R_∞ is not invertible. Controllability in finite time near $\tilde z$ is obtained combining the results of [8] and [10]: there is a constant $\delta > 0$ and a time $T > 0$ such that for any $z_0, z_1 \in S \cap B_{H^3_{(V)}}(\tilde z,\delta)$ there is a control $v \in L^2([0,T],\mathbb{R})$ verifying $\mathcal{U}_T(z_0,v) = z_1$. Let us prove that the problem is exactly controllable in infinite time in $S \cap B_{H^3_{(V)}}(\tilde z,\delta)$. Take any $z_1 \in S \cap B_{H^3_{(V)}}(\tilde z,\delta)$ and let us show that there is a control $u \in \Theta$ such that $z_1 \in \mathcal{U}_\infty(\tilde z,u)$. Let us suppose first that $z_1 \notin \mathcal{E}$. Then, by Case 1, there is $\delta_{z_1} > 0$ such that exact controllability in infinite time holds in $S \cap B_{H^3_{(V)}}(z_1,\delta_{z_1})$. By exact controllability property in finite time and by an approximation argument, one can find a control $u_1 \in C_0^\infty((0,T),\mathbb{R})$ such that $\mathcal{U}_T(\tilde z,u_1) \in B_{H^3_{(V)}}(z_1,\delta_{z_1})$. Thus the existence of u_1 follows from Case 1 and Remark 3.11.

Now let us suppose that $z_1 \in \mathcal{E}$. Since $\mathcal{E} \subset \bigcap_{k=1}^\infty H^k_{(V)}$, by [8] and [10], there is a control $u_1 \in C^s([0,T],\mathbb{R})$ such that $\mathcal{U}_T(\tilde z,u_1) = z_1$ and $u(0) = \cdots = u^{(s)}(0) = u(T) = \cdots = u^{(s)}(T) = 0$. Extending u_1 by 0 on $[T,+\infty)$, we obtain $z_1 \in \mathcal{U}_\infty(\tilde z,u_1)$. □

Proof of Proposition 3.10. It suffices to show that for any $\varepsilon > 0$ there exists $\delta > 0$ for which

$$e\big(\tilde{\mathcal{U}}_\infty(z_0,u) - R_\infty(z_0,u), \tilde{\mathcal{U}}_\infty\big(z_0',u'\big) - R_\infty\big(z_0',u'\big)\big) \leqslant \varepsilon \big\|(z_0,u) - \big(z_0',u'\big)\big\|_{T_{\tilde z}\cap H^3_{(V)}\times\Theta}, \tag{3.16}$$

whenever $(z_0,u), (z_0',u') \in B_{T_{\tilde z}\cap H^3_{(V)}\times\Theta}((0,0),\delta)$. Here $e(\cdot,\cdot)$ stands for the Hausdorff distance (see Appendix A for the definition). It is clear from the definition of $e(\cdot,\cdot)$, that (3.16) follows from the following stronger estimate

$$\sup_{t\in\mathbb{R}_+} \big\|\mathcal{U}_t\big(P^{-1}z_0,u\big) - R_t(z_0,u) - \mathcal{U}_t\big(P^{-1}z_0',u'\big) + R_t\big(z_0',u'\big)\big\|_{T_{\tilde z}\cap H^3_{(V)}} \leqslant \varepsilon \big\|(z_0,u) - \big(z_0',u'\big)\big\|_{T_{\tilde z}\cap H^3_{(V)}\times\Theta}.$$

To prove this estimate, notice that the function

$$y(t) := \mathcal{U}_t\big(P^{-1}z_0,u\big) - R_t(z_0,u) - \mathcal{U}_t\big(P^{-1}z_0',u'\big) + R_t\big(z_0',u'\big)$$

is a solution of the problem

$$\begin{aligned} i\dot{y} &= -\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + (u-u')Q\big(\mathcal{U}_t\big(P^{-1}z_0,u\big)-\mathcal{U}_t(\tilde{z},0)\big) + u'Q\big(\mathcal{U}_t\big(P^{-1}z_0,u\big)-\mathcal{U}_t\big(P^{-1}z_0',u\big)\big),\\ y|_{\partial D} &= 0,\\ y(0,x) &= P^{-1}z_0 - z_0 - P^{-1}z_0' + z_0'. \end{aligned}$$

We have

$$\big\|y(0)\big\|_{3,V} \leqslant \varepsilon \|z_0 - z_0'\|_{3,V} \tag{3.17}$$

for any $z_0, z_0' \in B_{T_{\tilde{z}}\cap H^3_{(V)}}(0,\delta)$ and for sufficiently small $\delta > 0$. Using (3.4) (we use the version of the inequality with $v_1 f_1 + v_2 f_2$ instead of vf), Corollary 3.3 and (3.17), we get

$$\begin{aligned} \sup_{t\in\mathbb{R}_+}\big\|y(t)\big\|_{3,V} &\leqslant C\Big(\big\|y(0)\big\|_{3,V} + \sup_{t\in\mathbb{R}_+}\big\|\mathcal{U}_t\big(P^{-1}z_0,u\big)-\mathcal{U}_t(\tilde{z},0)\big\|_{3,V}\big\|u-u'\big\|_\Theta \\ &\quad + \sup_{t\in\mathbb{R}_+}\big\|\mathcal{U}_t\big(P^{-1}z_0,u\big)-\mathcal{U}_t\big(P^{-1}z_0',u\big)\big\|_{3,V}\big\|u'\big\|_\Theta\Big) \\ &\leqslant C\big(\big\|y(0)\big\|_{3,V} + \big(\|z_0\|_{3,V}+\|u\|_\Theta\big)\big\|u-u'\big\|_\Theta + \big\|z_0-z_0'\big\|_{3,V}\big\|u'\big\|_\Theta\big) \\ &\leqslant \varepsilon\big\|(z_0,u)-\big(z_0',u'\big)\big\|_{T_{\tilde{z}}\cap H^3_{(V)}\times\Theta} \end{aligned}$$

for sufficiently small δ. This proves the proposition. □

4. Non-controllability result

4.1. Main result

In this section, we study the problem of non-controllability of Schrödinger system (1.1)–(1.3), where $D \subset \mathbb{R}^d$ is a bounded domain with smooth boundary, $V, Q \in C^\infty(\overline{D},\mathbb{R})$ are arbitrary given functions. The following lemma establishes the well-posedness of system (1.1)–(1.3) in the space L^2.

Lemma 4.1. *For any $z_0 \in L^2$ and for any $u \in L^1_{\mathrm{loc}}(\mathbb{R}_+,\mathbb{R})$, problem* (1.1)–(1.3) *has a unique solution $z \in C(\mathbb{R}_+, L^2)$. Furthermore, the resolving operator $\mathcal{U}_t(\cdot,u): L^2 \to L^2$ taking z_0 to $z(t)$ satisfies the relation*

$$\big\|\mathcal{U}_t(z_0,u)\big\| = \|z_0\|, \quad t \geqslant 0.$$

See [11] for the proof. Let us define the set of attainability of system (1.1), (1.2) from an initial point $z_0 \in S$:

$$\mathcal{A}(z_0) := \big\{\mathcal{U}_t(z_0,u)\text{: for all } u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+,\mathbb{R}) \text{ and } t \geqslant 0\big\}. \tag{4.1}$$

The following theorem is the main result of this section.

Theorem 4.2. *For any constant $k \in (0,d)$, any initial condition $z_0 \in S$ and any ball $B \subset H^k_{(V)}$, we have*

$$\mathcal{A}^c(z_0) \cap B \cap S \neq \varnothing.$$

Let us emphasize that this theorem does not exclude exact controllability in $H^k_{(V)}$ with controls form a larger space than $W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+,\mathbb{R})$.

The proof of this theorem is an adaptation of ideas of Shirikyan [28] to the case of Schrödinger equation. Using a Hölder type estimate for the solution of the equation, we show that the image by the resolving operator $\mathcal{U}$ of a ball in the space of controls has a Kolmogorov ε-entropy strictly less than that of a ball B in the phase space $H^k_{(V)}$. As we show, this implies the non-controllability.

4.2. Some ε-entropy estimates

Let X be a Banach space. For any compact set $K \subset X$ and $\varepsilon > 0$, we denote by $N_\varepsilon(K, X)$ the minimal number of sets of diameters $\leqslant 2\varepsilon$ that are needed to cover K. The Kolmogorov ε-entropy of K is defined as $H_\varepsilon(K, X) = \ln N_\varepsilon(K, X)$.

Let Y be another Banach space and let $f : K \to Y$ be a Hölder continuous function:

$$\big\| f(u_1) - f(u_2)\big\|_Y \leqslant L\|u_1 - u_2\|_X^\theta \tag{4.2}$$

for any $u_1, u_2 \in K$ and for some constants $L > 0$ and $\theta \in (0, 1)$. The following lemma follows immediately from the definition of ε-entropy (cf. Lemma 2.1 in [28]).

Lemma 4.3. *For any compact set $K \subset X$ and any function $f : K \to Y$ satisfying inequality* (4.2)*, we have*

$$H_\varepsilon\big(f(K), Y\big) \leqslant H_{(\frac{\varepsilon}{L})^{\frac{1}{\theta}}}(K, X) \quad \text{for all } \varepsilon > 0.$$

We also need the following two lemmas.

Lemma 4.4. *For any $T > 0$ and for any closed ball $B \subset W^{1,1}([0, T], \mathbb{R})$, there is a constant $C > 0$ such that*

$$H_\varepsilon\big(B, L^1\big([0, T], \mathbb{R}\big)\big) \leqslant \frac{C}{\varepsilon} \ln \frac{1}{\varepsilon}.$$

This is Proposition 2.3 in [28].

Lemma 4.5. *For any $k > 0$ and any closed ball $B := \overline{B_{H^k_{(V)}}(z_0, r)}$ such that $B_{H^k_{(V)}}(z_0, r) \cap S \neq \varnothing$ there is a constant $C > 0$ such that*

$$H_\varepsilon\big(B \cap S, H^{k-1}\big) \geqslant C\left(\frac{1}{\varepsilon}\right)^d. \tag{4.3}$$

Proof. It is well known that

$$C_1\left(\frac{1}{\varepsilon}\right)^d \leqslant H_\varepsilon\big(B, H^{k-1}\big) \leqslant C_2\left(\frac{1}{\varepsilon}\right)^d \tag{4.4}$$

for some constants $C_1, C_2 > 0$ (e.g., see [14]). Consider the mapping

$$f : \left[\frac{1}{2}, \frac{3}{2}\right] \times B \cap S \to H^{k-1},$$
$$(s, z) \to sz.$$

The set $f([\frac{1}{2}, \frac{3}{2}] \times B \cap S)$ has a non-empty interior, so there is a ball $\tilde{B}$ in H^k such that

$$\tilde{B} \subset f\left(\left[\frac{1}{2}, \frac{3}{2}\right] \times B \cap S\right). \tag{4.5}$$

Clearly,

$$\big\| f(s_1, z_1) - f(s_2, z_2)\big\|_{k-1} \leqslant C\big(|s_1 - s_2| + \|z_1 - z_2\|_{k-1}\big).$$

Using (4.5) and Lemma 4.3, we get

$$\begin{aligned} H_\varepsilon\big(\tilde{B}, H^{k-1}\big) &\leqslant H_\varepsilon\left(f\left(\left[\frac{1}{2}, \frac{3}{2}\right] \times B \cap S\right), H^{k-1}\right) \leqslant H_{\frac{\varepsilon}{C}}\left(\left[\frac{1}{2}, \frac{3}{2}\right] \times B \cap S, \mathbb{R} \times H^{k-1}\right) \\ &\leqslant H_{\frac{\varepsilon}{C}}\left(\left[\frac{1}{2}, \frac{3}{2}\right], \mathbb{R}\right) + H_{\frac{\varepsilon}{C}}\big(B \cap S, H^{k-1}\big) \leqslant C\left(\ln\frac{1}{\varepsilon} + H_\varepsilon\big(B \cap S, H^{k-1}\big)\right). \end{aligned}$$

Combining this with (4.4) for $\tilde{B}$, we obtain (4.3). □

4.3. Proof of Theorem 4.2

Let us suppose, by contradiction, that there is $k \in (0, d)$, an initial point $z_0 \in S$ and a ball $B \subset H^k_{(V)}$ such that

$$B \cap S \subset \mathcal{A}(z_0), \tag{4.6}$$

where $\mathcal{A}$ is the set of attainability of system (1.1), (1.2) from the initial point z_0 defined by (4.1). Let us set

$$B_m := [0, m] \times B_{W^{1,1}([0,m],\mathbb{R})}(0, m),$$
$$\mathcal{U}(B_m) := \{\mathcal{U}_t(z_0, u)\colon \text{ for all } (t, u) \in B_m\}.$$

We have

$$\mathbb{R} \times W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}) = \bigcup_{m=1}^{\infty} B_m,$$
$$\mathcal{A}(z_0) = \bigcup_{m=1}^{\infty} \mathcal{U}(B_m). \tag{4.7}$$

Combining (4.6), (4.7) and the Baire lemma, we see that there is a ball $Q \subset H^k_{(V)}$ and an integer $m \geqslant 1$ such that $\mathcal{U}(B_m)$ is dense in $Q \cap S$ with respect to H^k-norm.

Step 1. Let us define the set

$$\tilde{B}_m = \{(t, u) \in B_m\colon \text{ such that } \mathcal{U}_t(z_0, u) \in Q\}.$$

Here we prove that $\tilde{B}_m$ is compact in $[0, m] \times L^1([0, m], \mathbb{R})$. Indeed, take any sequence $(t_n, u_n) \in \tilde{B}_m$. As $(t_n, u_n) \in B_m$ and B_m is compact in $[0, m] \times L^1([0, m], \mathbb{R})$, there is a sequence $n_k \to \infty$ and $(t_0, u_0) \in B_m$ such that

$$|t_{n_k} - t_0| + \|u_{n_k} - u_0\|_{L^1([0,m],\mathbb{R})} \to 0, \quad k \to \infty.$$

We need to show that $(t_0, u_0) \in \tilde{B}_m$. As $\mathcal{U}_{t_{n_k}}(z_0, u_{n_k}) \in Q$, there is $z \in Q$ such that $\mathcal{U}_{t_{n_k}}(z_0, u_{n_k}) \rightharpoonup z$ in H^k (again extracting a subsequence, if necessary). On the other hand, Lemma 4.1 implies that $\mathcal{U}_{t_{n_k}}(z_0, u_{n_k}) \to \mathcal{U}_{t_0}(z_0, u_0)$ in L^2. Thus $\mathcal{U}_{t_0}(z_0, u_0) = z$ and $(t_0, u_0) \in \tilde{B}_m$. Thus $\tilde{B}_m$ is compact in $[0, m] \times L^1([0, m], \mathbb{R})$.

In particular, this implies that $\mathcal{U}(\tilde{B}_m)$ is compact in L^2, as an image of a compact set by a continuous mapping. On the other hand, $\mathcal{U}(\tilde{B}_m)$ is dense in the compact set $Q \cap S$ in L^2. Thus $Q \cap S = \mathcal{U}(\tilde{B}_m)$.

Step 2. Using standard arguments, one can show that we have

$$\|\mathcal{U}_t(z_0, u) - \mathcal{U}_{t'}(z_0, u')\| \leqslant C(|t - t'| + \|u - u'\|_{L^1([0,m],\mathbb{R})})$$

for any $(t, u), (t', u') \in \tilde{B}_m$, where $C > 0$ is a constant not depending on (t, u) and (t', u'). Combining this with the interpolation inequality

$$\|z\|_{k-1} \leqslant C\|z\|^{\frac{1}{k}}\|z\|_k^{\frac{k-1}{k}},$$

we get

$$\|\mathcal{U}_t(z_0, u) - \mathcal{U}_{t'}(z_0, u')\|_{k-1} \leqslant C(|t - t'|^{\frac{1}{k}} + \|u - u'\|^{\frac{1}{k}}_{L^1([0,m],\mathbb{R})})$$

for any $(t, u), (t', u') \in \tilde{B}_m$. Here we used the fact that $\mathcal{U}_t(z_0, u), \mathcal{U}_{t'}(z_0, u') \in Q$. Applying Lemmas 4.3 and 4.4 and the fact that $Q \cap S \subset \mathcal{U}(\tilde{B}_m)$, we obtain

$$H_\varepsilon(Q \cap S, H^{k-1}) \leqslant H_\varepsilon(\mathcal{U}(\tilde{B}_m), H^{k-1}) \leqslant C H_{\varepsilon^k}(\tilde{B}_m, [0, m] \times L^1([0, m], \mathbb{R}))$$
$$\leqslant C H_{\varepsilon^k}(B_m, [0, m] \times L^1([0, m], \mathbb{R})) \leqslant \frac{C}{\varepsilon^k} \ln \frac{1}{\varepsilon^k}.$$

This estimate contradicts Lemma 4.5 and proves the theorem.

Remark 4.6. The same proof works also in the case of Schrödinger equation with any finite number of controls:

$$i\dot{z} = -\Delta z + V(x)z + u_1(t)Q_1(x)z + \cdots + u_n(t)Q_n(x)z,$$

where $n \geqslant 1$ is any integer, $Q_j \in C^\infty(\overline{D}, \mathbb{R})$ are arbitrary functions and u_j are the controls $j = 1, \dots, n$.

Acknowledgements

The authors would like to thank Armen Shirikyan for many fruitful conversations.

Appendix A

A.1. Genericity of Condition 2.5

Let us assume that $D = (0,1)^d$ and introduce the space

$$\mathcal{G} := \big\{V \in C^\infty(D, \mathbb{R}):\ V(x_1, \dots, x_d) = V_1(x_1) + \cdots + V_d(x_d) \text{ for some } V_k \in C^\infty\big([0,1], \mathbb{R}\big), k = 1, \dots, d\big\}.$$

Then $\mathcal{G}$, endowed with the metric of $C^\infty(\overline{D}, \mathbb{R})$, is a closed subspace in $C^\infty(\overline{D}, \mathbb{R})$. By Lemma 3.12 in [22], the set $\mathcal{A}$ of all functions $V \in \mathcal{G}$ such that property (ii) in Condition 2.5 is verified is G_δ set (i.e., countable intersection of dense open sets). First let us prove genericity of property (i) in the case $d = 1$.

Lemma A.1. *For any $V \in C^\infty([0,1], \mathbb{R})$, the set of functions $Q \in C^\infty([0,1], \mathbb{R})$ such that*

$$\inf_{p,j \geqslant 1} \big|p^3 j^3 \langle Q e_{p,V}, e_{j,V}\rangle\big| > 0 \tag{A.1}$$

is dense in $C^\infty([0,1], \mathbb{R})$.

Proof. If $V = 0$, then a straightforward calculation gives

$$\big\langle x^2 e_{p,0}, e_{j,0}\big\rangle = \begin{cases} \frac{(-1)^{p+j} 8pj}{\pi^2(p^2-j^2)^2}, & \text{if } p \neq j, \\ \frac{2}{3} - \frac{1}{p^2\pi^2}, & \text{if } p = j, \end{cases}$$

which implies (A.1) for $Q = x^2$ and $V = 0$. In the general case, taking any $p \neq j$, we integrate by parts (we write λ_j, e_j and z'', z' instead of $\lambda_{j,V}, e_{j,V}$ and $\frac{\mathrm{d}^2 z}{\mathrm{d}x^2}, \frac{\mathrm{d}z}{\mathrm{d}x}$, respectively)

$$\langle Q e_p, e_j\rangle = \frac{1}{\lambda_j}\left\langle\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V\right)(Q e_p), e_j\right\rangle = \frac{1}{\lambda_j}\big(\langle -Q'' e_p, e_j\rangle + \big\langle -Q' e_p', e_j\big\rangle + \lambda_p\langle Q e_p, e_j\rangle\big).$$

This implies that

$$\langle Q e_p, e_j\rangle = -\frac{1}{\lambda_j - \lambda_p}\big(\langle Q'' e_p, e_j\rangle + \big\langle Q' e_p', e_j\big\rangle\big). \tag{A.2}$$

Again integrating by parts, we get

$$\big\langle Q' e_p', e_j\big\rangle = \frac{1}{\lambda_j}\left\langle Q' e_p', \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V\right)e_j\right\rangle = -\frac{1}{\lambda_j} Q' e_p' e_j'\Big|_{x=0}^{x=1} + \frac{1}{\lambda_j}\left\langle\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V\right)\big(Q' e_p'\big), e_j\right\rangle. \tag{A.3}$$

Notice that

$$\left\langle\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V\right)\big(Q' e_p'\big), e_j\right\rangle = \big\langle V Q' e_p', e_j\big\rangle + \big\langle -Q''' e_p', e_j\big\rangle + \big\langle -Q'' e_p'', e_j\big\rangle + \lambda_p\big\langle Q' e_p', e_j\big\rangle - \big\langle Q'(V e_p)', e_j\big\rangle.$$

Replacing this into (A.3), we get

$$\big\langle Q' e_p', e_j\big\rangle = \frac{1}{\lambda_j - \lambda_p}\big(-Q' e_p' e_j'\big|_{x=0}^{x=1} + \big\langle V Q' e_p', e_j\big\rangle + \big\langle -Q''' e_p', e_j\big\rangle + \big\langle -Q'' e_p'', e_j\big\rangle - \big\langle Q'(V e_p)', e_j\big\rangle\big). \tag{A.4}$$

Using (A.2) and (A.4) and the fact that

$$\big\langle -Q''e''_p, e_j\big\rangle = -\big\langle Q''Ve_p, e_j\big\rangle + \lambda_p\big\langle Q''e_p, e_j\big\rangle,$$

we obtain

$$\begin{aligned}\langle Qe_p, e_j\rangle &= \left(-\frac{1}{\lambda_j-\lambda_p}\big\langle Q''e_p, e_j\big\rangle - \frac{\lambda_p}{(\lambda_j-\lambda_p)^2}\big\langle Q''e_p, e_j\big\rangle\right) - \frac{1}{(\lambda_j-\lambda_p)^2}\big(-Q'e'_pe'_j\big|_{x=0}^{x=1} + \big\langle VQ'e'_p, e_j\big\rangle \\ &\quad + \big\langle -Q'''e'_p, e_j\big\rangle - \big\langle Q''Ve_p, e_j\big\rangle - \big\langle Q'(Ve_{p,})', e_j\big\rangle\big) \\ &=: I_1 + I_2.\end{aligned}$$

Let Q be such that $A := Q'(x)\cos(p\pi x)\cos(j\pi x)|_{x=0}^{x=1} \neq 0$. Clearly, this is verified for almost any Q, since A depends only on the parity of p and j. Let us choose Q such that $\langle Qe_p, e_j\rangle \neq 0$ for all $p, j \geqslant 1$; the set of such functions Q is G_δ, by Section 3.4 in [22]. Using the estimates (1.7)–(1.9), it is easy to see that $\inf_{p,j\geqslant 1, p\neq j} |p^3j^3I_2| > 0$. Iterating the same arguments for I_1, we see that $\inf_{p,j\geqslant 1, p\neq j} |p^3j^3\langle Qe_{p,V}, e_{j,V}\rangle| > 0$ for almost any polynomial Q.

If $p = j$, using (1.8), we get

$$\langle Qe_p, e_p\rangle = 2\big\langle Q, \sin^2(p\pi x)\big\rangle + s_p,$$

where $s_p \to 0$. Thus

$$\langle Qe_p, e_p\rangle = \langle Q, 1-\cos 2p\pi x\rangle + s_p = \int_0^1 Q\,\mathrm{d}x - \langle Q, \cos 2p\pi x\rangle + s_p.$$

Taking Q such that $\int_0^1 Q\,\mathrm{d}x \neq 0$, we complete the proof of the lemma. □

Take any functions $Q_k \in C^\infty([0,1], \mathbb{R})$, $k = 1, \dots, d$, in the dense set of Lemma A.1 corresponding to $V_k \in C^\infty([0,1], \mathbb{R})$, $k = 1, \dots, d$. Then $Q(x_1, \dots, x_d) := Q_1(x_1)\cdot \dots \cdot Q_d(x_d)$ satisfies (i) with $V(x_1, \dots, x_d) := V_1(x_1) + \dots + V_d(x_d)$.

A.2. Inverse mapping theorem for multifunctions

In this section, we recall the statement of the inverse mapping theorem for multivalued functions or multifunctions. We refer the reader to the paper [21] by Nachi and Penot for details and for a review of the literature on this subject.

Let X and Y be Banach spaces. For any non-empty sets $C, D \subset X$, define the Hausdorff distance

$$d(x, D) = \inf_{y\in D} \|x - y\|_X,$$
$$e(C, D) = \sup_{x\in C} d(x, D).$$

We call a multifunction from X to Y any mapping F from X to 2^Y.

Definition A.2. A multifunction F from an open set $X_0 \subset X$ to Y is said to be strictly differentiable at (x_0, y_0) if there exists some continuous linear map $A : X \to Y$ such that for any $\varepsilon > 0$ there exist $\beta, \delta > 0$ for which

$$e\big(F(x) \cap B_Y(y_0, \beta) - A(x), F\big(x'\big) - A\big(x'\big)\big) \leqslant \varepsilon\big\|x - x'\big\|_X,$$

whenever $x, x' \in B(x_0, \delta)$. The map A is called a derivative of F at (x_0, y_0).

The following theorem is a generalization of the classical inverse function theorem to the case of multifunctions.

Theorem A.3. *Let F be a multifunction from an open set $X_0 \subset X$ to Y with closed non-empty values. Suppose F is strictly differentiable at $(x_0, y_0) \in Gr(F)$, and some derivative A of F at (x_0, y_0) has a right inverse. Then for any neighborhood U of x_0 there exists a neighborhood V of y_0 such that $V \subset F(U)$.*

See Theorem 22 in [21] for the proof.

References

[1] A. Agrachev, T. Chambrion, An estimation of the controllability time for single-input systems on compact Lie groups, ESAIM Control Optim. Calc. Var. 12 (3) (2006) 409–441.
[2] F. Albertini, D. D'Alessandro, Notions of controllability for bilinear multilevel quantum systems, IEEE Trans. Automat. Control 48 (8) (2003) 1399–1403.
[3] C. Altafini, Controllability of quantum mechanical systems by root space decomposition of $su(n)$, J. Math. Phys. 43 (5) (2002) 2051–2062.
[4] J.M. Ball, J.E. Marsden, M. Slemrod, Controllability for distributed bilinear systems, SIAM J. Control Optim. 20 (4) (1982) 575–597.
[5] L. Baudouin, J.-P. Puel, Uniqueness and stability in an inverse problem for the Schrödinger equation, Inverse Problems 18 (6) (2001) 1537–1554.
[6] K. Beauchard, Local controllability of a 1D Schrödinger equation, J. Math. Pures Appl. 84 (7) (2005) 851–956.
[7] K. Beauchard, Y. Chitour, D. Kateb, R. Long, Spectral controllability of 2D and 3D linear Schrödinger equations, J. Funct. Anal. 256 (2009) 3916–3976.
[8] K. Beauchard, J.-M. Coron, Controllability of a quantum particle in a moving potential well, J. Funct. Anal. 232 (2) (2006) 328–389.
[9] K. Beauchard, J.-M. Coron, M. Mirrahimi, P. Rouchon, Implicit Lyapunov control of finite dimensional Schrödinger equations, Syst. Contr. Lett. 56 (5) (2007) 388–395.
[10] K. Beauchard, C. Laurent, Local controllability of linear and nonlinear Schrödinger equations with bilinear control, Preprint, 2010.
[11] T. Cazenave, Semilinear Schrödinger Equations, Courant Lecture Notes in Mathematics, vol. 10, AMS, 2003.
[12] T. Chambrion, P. Mason, M. Sigalotti, U. Boscain, Controllability of the discrete-spectrum Schrödinger equation driven by an external field, Ann. IHP Anal. Non Lin. 26 (1) (2009) 329–349.
[13] B. Dehman, P. Gérard, G. Lebeau, Stabilization and control for the nonlinear Schrödinger equation on a compact surface, Math. Z. 254 (4) (2006) 729–749.
[14] D.E. Edmunds, H. Triebel, Function Spaces, Entropy Numbers, Differential Operators, Cambridge University Press, Cambridge, 1996.
[15] S. Ervedoza, J.-P. Puel, Approximate controllability for a system of Schrödinger equations modeling a single trapped ion, Ann. IHP Anal. Non Lin. 26 (6) (2009) 2111–2136.
[16] A. Katok, B. Hasselblatt, Introduction to the Modern Theory of Dynamical Systems, Encyclopedia of Mathematics and its Applications, vol. 54, Cambridge University Press, Cambridge, 1995.
[17] G. Lebeau, Contrôle de l'équation de Schrödinger, J. Math. Pures Appl. 71 (3) (1992) 267–291.
[18] E. Machtyngier, E. Zuazua, Stabilization of the Schrödinger equation, Portugaliae Math. 51 (2) (1994) 243–256.
[19] P. Mason, M. Sigalotti, Generic controllability properties for the bilinear Schrödinger equation, Comm. in PDE 35 (4) (2010) 685–706.
[20] M. Mirrahimi, Lyapunov control of a particle in a finite quantum potential well, in: IEEE Conf. on Decision and Control, San Diego, 2006.
[21] K. Nachi, J.-P. Penot, Inversion of multifunctions and differential inclusions, Control Cybernet. 34 (3) (2005) 871–901.
[22] V. Nersesyan, Growth of Sobolev norms and controllability of the Schrödinger equation, Comm. Math. Phys. 290 (1) (2009) 371–387.
[23] V. Nersesyan, Global approximate controllability for Schrödinger equation in higher Sobolev norms and applications, Ann. IHP Anal. Non Lin. 27 (3) (2010) 901–915.
[24] H. Nersisyan, Controllability of 3D incompressible Euler equations by a finite-dimensional external force, ESAIM: COCV 16 (2010) 677–694, doi:10.1051/cocv/2009017.
[25] J. Pöschel, E. Trubowitz, Inverse Spectral Theory, Academic Press, New York, 1987.
[26] Y. Privat, M. Sigalotti, The squares of Laplacian–Dirichlet eigenfunctions are generically linearly independent, Preprint, 2008.
[27] V. Ramakrishna, M. Salapaka, M. Dahleh, H. Rabitz, A. Pierce, Controllability of molecular systems, Phys. Rev. A 51 (2) (1995) 960–966.
[28] A. Shirikyan, Euler equations are not exactly controllable by a finite-dimensional external force, Physica D 237 (2008) 1317–1323.
[29] G. Turinici, On the Controllability of Bilinear Quantum Systems, Lecture Notes in Chem., vol. 74, 2000.
[30] G. Turinici, H. Rabitz, Quantum wavefunction controllability, Chem. Phys. 267 (1) (2001) 1–9.
[31] E. Zuazua, Remarks on the controllability of the Schrödinger equation, CRM Proc. Lecture Notes 33 (2003) 193–211.