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Report on the PhD Thesis "Invertibilité restreinte, distance au cube et covariance de matrices aléatoires" by Pierre Youssef

It is my great pleasure to write a report on the excellent work of Pierre Youssef. His PhD Thesis contains deep and beautiful results on fundamental questions of the local theory of Banach spaces. He exploits an extremely interesting "linear algebraic method" which was recently discovered by Batson, Spielman and Srivastava in order to introduce new ideas in this area and to obtain beautiful simple proofs and extensions of results of Bourgain, Tzafriri and others, related to restricted invertibility, the Kadison-Singer problem and some classical problems about Banach-Mazur distance estimates.

The Thesis is divided in four Parts which I will attempt to describe briefly.

1. Restricted invertibility. The restricted invertibility principle of Bourgain and Tzafriri states that if T is an $n\times n$ matrix whose columns Te_j have Euclidean norm equal to 1 then there exists $\sigma\subset\{1,\ldots,n\}$ of cardinality $|\sigma|\geq cn/\|T\|^2$ such that, for any choice of scalars $(t_j)_{j\in\sigma}$,

$$\Big\|\sum_{j\in\sigma}t_jTe_j\Big\|_2\geq c\Big(\sum_{j\in\sigma}t_j^2\Big)^{1/2},$$

where $c>0$ is an absolute constant. In other words, the restriction T_σ of T to span$\{e_j : j\in\sigma\}$ is well-invertible; one has $s_{\min}(T_\sigma)\geq c$. Vershynin generalized this result as follows: if $Id=\sum_{j=1}^m x_jx_j^t$ and $T:\ell_2^n\to\ell_2^n$ is a linear operator then, for any $\varepsilon\in(0,1)$ one can find $\sigma\subset\{1,\ldots,m\}$ of cardinality $|\sigma|\geq(1-\varepsilon)\|T\|_{\rm HS}^2/\|T\|^2$ such that for any choice of scalars $(t_j)_{j\in\sigma}$,

$$\Big\|\sum_{j\in\sigma}t_j\frac{Te_j}{\|Te_j\|_2}\Big\|_2\geq c(\varepsilon)\Big(\sum_{j\in\sigma}t_j^2\Big)^{1/2},$$

where $c(\varepsilon)>0$ is a constant depending only on ε. Vershynin's argument is based on an iteration of the Bourgain-Tzafriri theorem and a result of Kashin-Tzafriri, and this affects the final dependence of $c(\varepsilon)$ on ε. One should also emphasize that the arguments behind both results involve a mixture of probabilistic, combinatorial and functional analytic methods.

The starting point of Youssef's approach is a restricted invertibility theorem of Spielman and Srivastava. They showed that if $x_1,\ldots,x_m\in\mathbb{R}^n$ satisfy $Id=\sum_{j=1}^m x_jx_j^t$ then, for every

$\varepsilon \in (0,1)$ and every linear operator $T : \ell_2^n \to \ell_2^n$ one can find $\sigma \subset \{1,\dots,m\}$ of cardinality $|\sigma| \geq (1-\varepsilon)^2\|T\|_{\mathrm{HS}}^2/\|T\|^2$ such that $(Tx_j)_{j\in\sigma}$ is linearly independent and, for any choice of scalars $(t_j)_{j\in\sigma}$,

$$\lambda_{\min}\left(\sum_{j\in\sigma}(Tx_j)(Tx_j)^t\right) \geq \frac{\varepsilon^2\|T\|_{\mathrm{HS}}^2}{m}.$$

The elementary proof of this fact has its origin in previous work of Batson, Spielman and Srivastava. It introduces a very interesting linear algebraic method and gives a deterministic algorithm for the selection of the vectors Tx_j.

The main result of Youssef in this Part of his Thesis is a normalized version of the restricted invertibility principle of Spielman and Srivastava which reads as follows: Let U be an $n \times m$ matrix and let $D = \mathrm{diag}(\alpha_1,\dots,\alpha_m)$ be a diagonal $m \times m$ matrix with the property that $\mathrm{Ker}(D) \subset \mathrm{Ker}(U)$. Then, for any $\varepsilon \in (0,1)$ there exists $\sigma \subset \{1,\dots,m\}$ with

$$|\sigma| \geq (1-\varepsilon)^2\frac{\|U\|_{\mathrm{HS}}^2}{\|U\|_2^2}$$

such that

$$s_{\min}\left(U_\sigma D_\sigma^{-1}\right) > \frac{\varepsilon\|U\|_{\mathrm{HS}}}{\|D\|_{\mathrm{HS}}}.$$

Equivalently, for any choice of scalars $(t_j)_{j\in\sigma}$ one has

$$\left|\sum_{j\in\sigma} t_j\frac{Ue_j}{\alpha_j}\right| \geq \varepsilon\frac{\|U\|_{\mathrm{HS}}}{\|D\|_{\mathrm{HS}}}\left(\sum_{j\in\sigma}t_j^2\right)^{1/2}.$$

The proof borrows ideas from the work of Spielman and Srivastava but the type of the statement obtained is exactly the one which is needed for an alternative proof of the Bourgain-Tzafriri and Vershynin results. In fact, Youssef improves both of them because (i) he is able to handle arbitrarily small values of ε in the Bourgain-Tzafriri result and (ii) he obtains a much better constant $c(\varepsilon)$ in Vershynin's result. In a few words, he presents an elementary and quantitatively better proof of these well-known results.

2. Proportional Dvoretzky-Rogers factorization and the Banach-Mazur distance to the cube. Youssef applies the main result of Part 1 to obtain a new proof of the best known quantitative estimate on the so-called proportional Dvoretzky-Rogers factorization: for every n-dimensional normed space X and any $\varepsilon > 0$ there exists $k \geq (1-\varepsilon)^2 n$ such that the identity $i_{2,\infty} : l_2^k \longrightarrow l_\infty^k$ can be written as $i_{2,\infty} = \alpha \circ \beta$, where $\beta : l_2^k \longrightarrow X$, $\alpha : X \longrightarrow l_\infty^k$ and $\|\alpha\| \cdot \|\beta\| \leq \frac{1}{\varepsilon}$. The previously known proof of this fact was rather complicated and involved the same combinatorial and functional analytic tools as the Bourgain-Tzafriri theorem (isomorphic versions of the Sauer-Shelah lemma and factorization theorems). The proof of Youssef is certainly simpler and direct, and hence, besides leading to the same dependence on ε, it has the advantage of providing better values for the absolute constants involved. It is also known that this result is a main ingredient in the available approach to the problem of the maximal Banach-Mazur distance from an n-dimensional normed space to ℓ_∞^n. This is a fascinating problem, posed by Pelczynski, which remains open. Youssef recovers the best known upper bound $d(X,\ell_\infty^n) \leq Cn^{5/6}$ and obtains a better value for the numerical constant C.

A proportional Dvoretzky-Rogers factorization result is also provided for not necessarily symmetric convex bodies in Löwner's position. This is an improved version (with respect to the dependence on ε) of a theorem of Litvak and Tomczak-Jaegermann.

3. Column subset selection. This part of the Thesis is related to the famous Kadison-Singer problem. Using the "linear algebraic method" instead of the probabilistic method of Bourgain and Tzafriri, he manages to obtain new simpler proofs of results of Tropp on partitioning a matrix with columns of norm one into blocks with singular values almost equal to 1 and he gives a new proof of the Bourgain-Tzafriri paving theorem: there exists an absolute constant $c > 0$ such that for every $\varepsilon \in (0,1)$ and any operator $T : \ell_2^n \to \ell_2^n$ with $\langle Te_i, e_i\rangle = 0$ for all $i \leq n$, there exists $\sigma \subseteq \{1,\dots,n\}$ satisfying $|\sigma| \geq c\varepsilon^2 n$ and $\|P_\sigma T P_\sigma^t\| \leq \varepsilon\|T\|$. The question to obtain a proof of this fact along these lines was posed by Naor.

4. Covariance of random matrices. The last Part of the Thesis is related to a well-known question of Kannan, Lovász and Simonovits concerning the minimal sample size $N = N(n)$ which is enough in order to estimate the covariance matrix of an n-dimensional distribution by the sample covariance matrix in the operator norm, with an arbitrary fixed accuracy. In the case of log-concave distributions, the optimal bound $N = O(n)$ was established by Adamczak, Litvak, Pajor and Tomczak-Jaegermann. The starting point of Youssef is subsequent work of Srivastava and Vershynin who, using randomization of the algorithm of Batson-Spielman-Srivastava, obtained the same result for every distribution whose k-dimensional marginals have uniformly bounded $2+\varepsilon$ moments outside a sphere of radius $C\sqrt{k}$. Following their approach, Youssef randomizes a method of De Carli Silva, Harvey and Sato and extends the result to the matrix setting.

Conclusion

The Thesis of Pierre Youssef is very well written and shows that he has a solid background in asymptotic geometric analysis. His work contains technical and beautiful results: a whole family of well-known results from the local theory of Banach spaces are proved by Youssef in an alternative, elementary and more direct way as consequences of his version of the Spielman-Srivastava restricted invertibility theorem. The estimates are equivalent to (and sometimes better than) the previously known ones.

From the above it should be clear that I find the work of Pierre Youssef particularly outstanding and I warmly recommend that his "thèse de doctorat" be approved with the highest distinction.

Sincerely,

AGiannopoulos

Apostolos Giannopoulos